8 Velocity Addition of Colliding Electrons

This chapter describes a thought experiment based on the inelastic collision of two electrons.

Two observers $\mathbf{O}_e$ and $\mathbf{O}_L$ in relative motion examine what is happening independently of one another. Both use the law of conservation of energy for their calculations. Then they compare their results using the rest mass $\boldsymbol{m_0}$ of the formed particle, which is invariant for both.

Thus, a single relation is formed from two equations, which represents the relative speed between the electrons as a function of the speeds of the single electrons.

With this method we will now derive the relativistic addition formula in the special case of equal velocities as follows.

We imagine the central collision between two unlike charged electrons (electron and positron) moving towards each other with equal velocities $\boldsymbol{v_e}$.[1] It is believed that the energy loss caused by the electromagnetic interaction between the electrons can be neglected because of the high speeds.

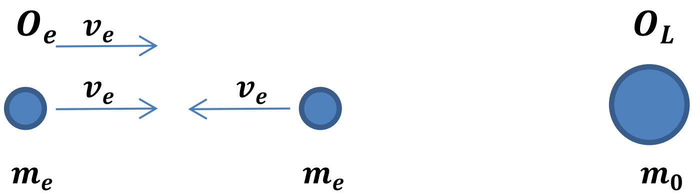

Fig. 9 (see the animation)

Suppose that because of the collision, a new particle of mass $\boldsymbol{m_0}$ is formed. This is from the perspective of the observer $\mathbf{O}_L$ at rest.

Assuming the observer $\mathbf{O}_L$ knows the velocities $\boldsymbol{v_e}$ of the electrons immediately before the collision, then he will be able to calculate the mass of the particle produced by the following equation which describes the conservation of energy before and after the collision:

$$\boldsymbol{m_e c^2 + m_e c^2 = m_0 c^2} \qquad (8.1)$$

That means:

The energy $\boldsymbol{m_0 c^2}$ of the particle produced is equal to the sum of the total energies of the colliding electrons.

Using the relation (5.4), equation (8.1) yields:

$$\boldsymbol{m_0 = \frac{2m_{0e}}{\sqrt{1-\frac{v_e^2}{c^2}}}} \qquad (8.2)$$

Where $\boldsymbol{m_0}$ and $\boldsymbol{m_{0e}}$ are the invariant masses of the formed particle and the electron.

[1] Such an experiment was already carried out in the 1960s in particle accelerators, so-called storage rings, and repeated countless times.

The observer $\mathbf{O}_L$ is thus able to calculate the mass of the particle produced and knows the velocities of the colliding electrons. However, he will not determine that the relative velocity $\boldsymbol{v}_{ee}$ between the electrons is equal to the sum of their velocities $\boldsymbol{v}_e + \boldsymbol{v}_e$, as is evident from the Galilean transformation only for low velocities, because that would not be in accordance with the conservation laws of energy and momentum.

To calculate the relative speed $\boldsymbol{v}_{ee}$ between the electrons, it is necessary to look at the same thought experiment from the point of view of a second observer $\mathbf{O}_e$ who is at rest to one of the electrons.

From the point of view of the observer $\mathbf{O}_e$ the following total energy $\boldsymbol{E}_1$ results before the collision:

$$\boldsymbol{E}_1 = \boldsymbol{m}_{0e}\boldsymbol{c}^2 + \boldsymbol{m}_{ee}\boldsymbol{c}^2 = \boldsymbol{m}_{0e}\boldsymbol{c}^2 + \frac{\boldsymbol{m}_{0e}\boldsymbol{c}^2}{\sqrt{1 - \frac{\boldsymbol{v}_{ee}^2}{\boldsymbol{c}^2}}} \qquad (8.3)$$

That is, the energy measured by $\mathbf{O}_e$ is equal to the sum of the energy of the resting electron and the energy of the other electron that the observer $\mathbf{O}_e$ is seeing approaching at the velocity $\boldsymbol{v}_{ee}$.

After the collision, observer $\mathbf{O}_e$, who continues to move at the velocity $\boldsymbol{v}_e$ to the formed particle, calculates its energy $\boldsymbol{E}_2$ by the following relation:

$$\boldsymbol{E}_2 = \frac{\boldsymbol{m}_0\boldsymbol{c}^2}{\sqrt{1 - \frac{\boldsymbol{v}_e^2}{\boldsymbol{c}^2}}} \qquad (8.4)$$

By replacing the invariant mass $\boldsymbol{m}_0$ by the expression (8.2) in equation (8.4) and setting $\boldsymbol{E}_1 = \boldsymbol{E}_2$ according to the law of conservation of energy, we get:

$$\boldsymbol{m}_{0e}\boldsymbol{c}^2 + \frac{\boldsymbol{m}_{0e}\boldsymbol{c}^2}{\sqrt{1 - \frac{\boldsymbol{v}_{ee}^2}{\boldsymbol{c}^2}}} = \frac{2\boldsymbol{m}_{0e}\boldsymbol{c}^2}{1 - \frac{\boldsymbol{v}_e^2}{\boldsymbol{c}^2}} \qquad (8.5)$$

Equation (8.5) can be further simplified by dividing all terms by $\boldsymbol{m}_{0e}\boldsymbol{c}^2$:

$$\frac{1}{\sqrt{1 - \frac{\boldsymbol{v}_{ee}^2}{\boldsymbol{c}^2}}} = \frac{2}{1 - \frac{\boldsymbol{v}_e^2}{\boldsymbol{c}^2}} - 1 \qquad \Rightarrow$$

$$\sqrt{1 - \frac{\boldsymbol{v}_{ee}^2}{\boldsymbol{c}^2}} = \frac{1 - \frac{\boldsymbol{v}_e^2}{\boldsymbol{c}^2}}{1 + \frac{\boldsymbol{v}_e^2}{\boldsymbol{c}^2}}$$

From this last expression, the following relation finally results after simple algebraic transformations:

$$\boldsymbol{v_{ee}} = \frac{\boldsymbol{2v_e}}{\boldsymbol{1 + \frac{v_e^2}{c^2}}} \tag{8.6}$$

Equation (8.6) represents the relative velocity between two particles that are moving towards each other at the same speeds and thus expresses the relativistic addition for the same speeds.

The same result is obtained if, instead of the law of conservation of energy, the law of conservation of momentum is used.

In this case, for the momentum before and after the collision, we get from the viewpoint of the observer $\boldsymbol{O_e}$:

$$\boldsymbol{m_{ee}v_{ee}} = \frac{\boldsymbol{m_0 v_e}}{\sqrt{\boldsymbol{1 - \frac{v_e^2}{c^2}}}}$$

And using (5.4) and (8.2):

$$\frac{\boldsymbol{m_{0e}v_{ee}}}{\sqrt{\boldsymbol{1 - \frac{v_{ee}^2}{c^2}}}} = \frac{\boldsymbol{2m_{0e}v_e}}{\boldsymbol{1 - \frac{v_e^2}{c^2}}}$$

This last equation, solved for $\boldsymbol{v_{ee}}$, yields the relation (8.6) (for details see AIII. in the Appendix).

We would like to stress at this point that the relation (8.6) was derived by the simple use of the law of conservation of energy and without the use of the axioms of special relativity.

The relation (8.6) shows that the relative speed at low velocities (e.g., in the case of two trains), i.e., when the term $\boldsymbol{\frac{v_e^2}{c^2}}$ is negligible, actually gives the sum of the velocities with good accuracy (green Curve in Fig. 10).

From the relation (8.6), however, the following general conclusion can be drawn:

Velocities cannot be added algebraically.

For the addition of equal velocities, the relation (8.6) can be used, which, in contrast to the simple algebraic addition, agrees with the laws of conservation of the energy and the momentum (violet curve).

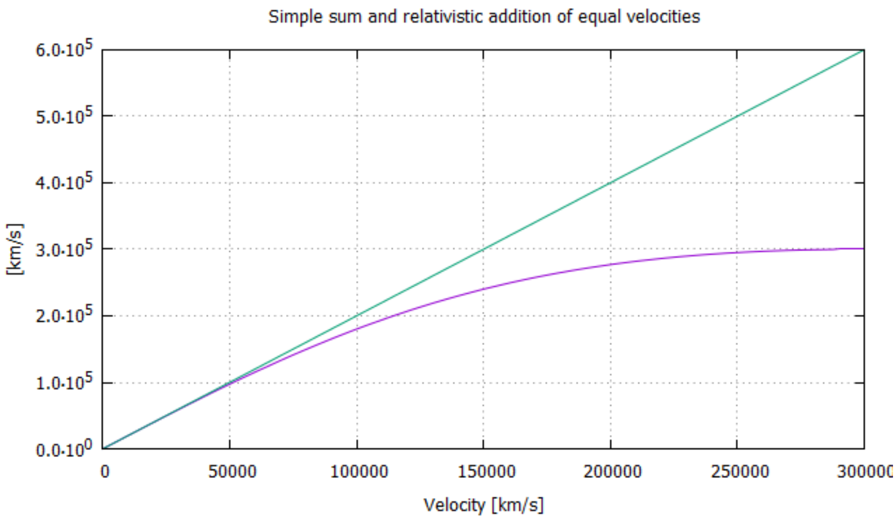

Fig. 10

It is easy to show that, according to the relation (8.6), the relative velocity $\boldsymbol{v_{ee}}$ between two particles can reach but not exceed the speed of light.

The relative velocity $\boldsymbol{v_{ee}}$ reaches the speed of light only if the colliding particles are massless and therefore $\boldsymbol{v_e}$ is equal to $\boldsymbol{c}$.

If it is assumed that the velocities of the colliding particles are different, an analogous derivation can be carried out for this case.

However, this is more complicated algebraically (see chapter 10).

If velocities $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are not equal, we will see that their addition leads to the following result:

$$\boldsymbol{v_{12} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}} \qquad (8.7)$$

The relation (8.7) is equivalent to Einstein's relativistic addition formula for velocities and reduces to (8.6) for $\boldsymbol{v_1 = v_2}$

From this it can be concluded that equation (8.6) represents a first confirmation of Einstein's formula (8.7) for <u>equal velocities</u>.

The application of the law of conservation of energy to the collision of two electrons represents a first confirmation of Einstein's relativistic addition formula for velocities.