10 The Velocity Addition Formula

The conventional methods for deriving the addition formula for velocities are based on transformations. In the fifth chapter, however, it was stated that in the further course of this treatise we have to forego the help of any transformation, because on the one hand the Galilean transformation can only be used for low speeds and on the other hand a transformation for any speed has not yet been derived. We are therefore only dependent on the use of the conservation laws for mass, energy and momentum.

To add the velocities, we differentiate between two cases: case 1. for low speeds and case 2. for any speeds. From case 1. it should become clear what kind of method is used.

We assume that because of the central collision of two particles $\boldsymbol{T_1}$ of mass $\boldsymbol{m_1}$ and $\boldsymbol{T_2}$ of mass $\boldsymbol{m_2}$, a researcher $\boldsymbol{O}$ observes the formation of a new particle $\boldsymbol{T}$ of mass $\boldsymbol{m_0}$, which remains with him at rest. A second observer $\boldsymbol{O_1}$ is in the state of rest for particle $\boldsymbol{T_1}$.

Case 1. Addition for low speeds

We imagine a physicist who would like to derive the addition formula of speed in the context of classical mechanics as an alternative, without using the Galilean transformation, which he does not know or whose correctness he does not trust. It refers to the thought experiment of Fig. 14 and only uses the conservation laws for mass and momentum of classical mechanics. For observer $\boldsymbol{O}$ it results:

$$\boldsymbol{m_0 = m_1 + m_2 \quad (a) \qquad and \qquad m_1 v_1 - m_2 v_2 = 0 \quad (b)}$$

If $\boldsymbol{v_{12}}$ is the speed of $\boldsymbol{T_2}$ from the point of view of observer $\boldsymbol{O_1}$, then the following applies for him:

$$\boldsymbol{m_2 v_{12} = m_0 v_1} \qquad (c)$$

Using relation (a) in (c) results in:

$$\boldsymbol{m_2 v_{12} = m_1 v_1 + m_2 v_1} \qquad (d)$$

From relation (b) we get: $\boldsymbol{m_1 v_1 = m_2 v_2}$ and that inserted into equation (d) results in:

$$\boldsymbol{v_{12} = v_1 + v_2}$$

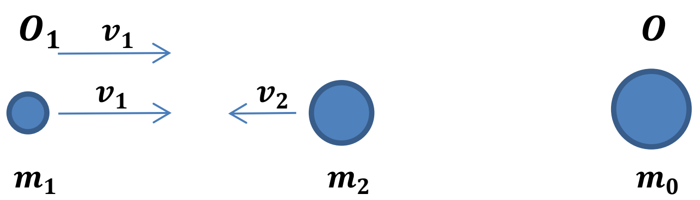

Fig. 14 (see the animation)

It will now be shown how the relativistic addition formula can be derived for any speed using an analogous method.

Case 2. Addition for any speed

In order to derive the relativistic addition formula for velocities, the energy and momentum conservation laws are applied to the collision of two dissimilar particles in the next thought experiment.

We will see that the somewhat more complicated calculation method, than that used in chapter 8 for equal masses and velocities, leads to the relation (8.7).

Assuming $\boldsymbol{m_{01}}$, $\boldsymbol{m_{02}}$ are the invariant masses and $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ are the velocities of $\boldsymbol{T_1}$ and $\boldsymbol{T_2}$, then the observer $\boldsymbol{O}$ will find the following relationship for the mass $\boldsymbol{m_0}$ of $\boldsymbol{T}$ due to the conservation of energy:

$$\boldsymbol{m_0 c^2 = m_1 c^2 + m_2 c^2} \qquad \Rightarrow$$

Or using the relation (5.4):

$$\boldsymbol{m_0 = \frac{m_{01}}{\sqrt{1-\frac{v_1^2}{c^2}}} + \frac{m_{02}}{\sqrt{1-\frac{v_2^2}{c^2}}}} \qquad (10.1)$$

On the other hand, the momentum of particle $\boldsymbol{T}$ is zero, because of momentum conservation before and after the collision:

$$\boldsymbol{\frac{m_{01}v_1}{\sqrt{1-\frac{v_1^2}{c^2}}} - \frac{m_{02}v_2}{\sqrt{1-\frac{v_2^2}{c^2}}} = 0} \qquad (10.2)$$

If $\boldsymbol{{}^{v_x}\!/_c}$ is replaced by $\boldsymbol{\beta_x}$ then:

$$\boldsymbol{\frac{m_{01}}{\sqrt{1-\beta_1^2}} = \frac{m_{02}}{\sqrt{1-\beta_2^2}}\frac{\beta_2}{\beta_1}} \qquad (10.3)$$

After the right-hand side of equation (10.3) has been inserted into equation (10.1) and the term $\boldsymbol{\frac{m_{02}}{\sqrt{1-\beta_2^2}}}$ has been excluded, the following results:

$$\boldsymbol{m_0 = \frac{m_{02}}{\sqrt{1-\beta_2^2}}\left(1+\frac{\beta_2}{\beta_1}\right)} \qquad (10.4)$$

Unlike (10.1), Equation (10.4) expresses the value of the invariant mass of the formed particle $\boldsymbol{T}$ as a function of only <u>one</u> of the masses of the colliding particles. We will use this interim result in the course of this derivation.

We now assume a second observer $\boldsymbol{O_1}$, who is at rest to the particle $\boldsymbol{T_1}$. This can calculate the relative velocity $\boldsymbol{v_{12}}$ between $\boldsymbol{T_1}$ and $\boldsymbol{T_2}$ by applying momentum conservation before and after collision of the particles as follows:

Since the observer $\boldsymbol{O_1}$ is at rest at $\boldsymbol{T_1}$, the momentum $\boldsymbol{p_1}$ measured by him before the collision is only that of the particle $\boldsymbol{T_2}$ with mass $\boldsymbol{m_{02}}$, which moves towards $\boldsymbol{O_1}$ with the velocity $\boldsymbol{v_{12}}$:

$$\boldsymbol{p_1} = \frac{\boldsymbol{m_{02} v_{12}}}{\sqrt{\boldsymbol{1 - \frac{v_{12}^2}{c^2}}}}$$

After the collision, $\boldsymbol{O_1}$ continues to move against particle $\boldsymbol{T}$ at the speed $\boldsymbol{v_1}$. Therefore, the momentum $\boldsymbol{p_2}$ measured by $\boldsymbol{O_1}$ is only that of the particle $\boldsymbol{T}$ with mass $\boldsymbol{m_0}$:

$$\boldsymbol{p_2} = \frac{\boldsymbol{m_0 v_1}}{\sqrt{\boldsymbol{1 - \frac{v_1^2}{c^2}}}}$$

Because of the law of conservation of momentum, the momentum before and after the collision must remain the same ($\boldsymbol{p_1} = \boldsymbol{p_2}$). It follows:

$$\frac{\boldsymbol{m_{02} v_{12}}}{\sqrt{\boldsymbol{1 - \frac{v_{12}^2}{c^2}}}} = \frac{\boldsymbol{m_0 v_1}}{\sqrt{\boldsymbol{1 - \frac{v_1^2}{c^2}}}} \qquad \Rightarrow$$

In order to be able to carry out the following calculations more easily, $\boldsymbol{v_x}/\boldsymbol{c}$ is now replaced by $\boldsymbol{\beta_x}$:

$$\frac{\boldsymbol{m_{02}\beta_{12}}}{\sqrt{\boldsymbol{1 - \beta_{12}^2}}} = \frac{\boldsymbol{m_0 \beta_1}}{\sqrt{\boldsymbol{1 - \beta_1^2}}} \qquad (10.5)$$

(From this point on, see also a derivation in Appendix A IV that is based on the law of conservation of energy).

The term for $\boldsymbol{m_0}$ from (10.4) is now inserted into equation (10.5):

$$\frac{\boldsymbol{m_{02}\ \beta_{12}}}{\sqrt{\boldsymbol{1 - \beta_{12}^2}}} = \frac{\boldsymbol{m_{02}}}{\sqrt{\boldsymbol{1 - \beta_1^2}}\sqrt{\boldsymbol{1 - \beta_2^2}}}\boldsymbol{\beta_1}\left(\boldsymbol{1 + \frac{\beta_2}{\beta_1}}\right) \qquad \Rightarrow$$

Now $\boldsymbol{m_{02}}$ can be shortened on both sides and the bracket with the factor $\boldsymbol{\beta_1}$ can be simplified:

$$\frac{\boldsymbol{\beta_{12}}}{\sqrt{\boldsymbol{1 - \beta_{12}^2}}} = \frac{\boldsymbol{\beta_1 + \beta_2}}{\sqrt{\boldsymbol{1 - \beta_1^2}}\sqrt{\boldsymbol{1 - \beta_2^2}}} \qquad \Rightarrow$$

To solve the roots, both terms are squared left and right:

$$\frac{\boldsymbol{\beta_{12}^2}}{\boldsymbol{1 - \beta_{12}^2}} = \frac{\boldsymbol{\beta_1^2 + 2\beta_1\beta_2 + \beta_2^2}}{\boldsymbol{1 - \beta_1^2 - \beta_2^2 + \beta_1^2\beta_2^2}} \qquad \Rightarrow$$

$$\boldsymbol{\beta_{12}^2 - \beta_{12}^2\beta_1^2 - \beta_{12}^2\beta_2^2 + \beta_{12}^2\beta_1^2\beta_2^2 =}$$
$$\boldsymbol{= \beta_1^2 + 2\beta_1\beta_2 + \beta_2^2 - \beta_{12}^2\beta_1^2 - 2\beta_{12}^2\beta_1\beta_2 - \beta_{12}^2\beta_2^2}$$

Finally, the terms $\boldsymbol{\beta_{12}^2\beta_1^2}$ and $\boldsymbol{\beta_{12}^2\beta_2^2}$, which occur with the same sign on both sides of the equation, are eliminated. In addition, the term $\boldsymbol{2\beta_{12}^2\beta_1\beta_2}$ is transferred from the right to the left side of the equation.

It follows:

$$\boldsymbol{\beta_{12}^2 + 2\beta_{12}^2\beta_1\beta_2 + \beta_{12}^2\beta_1^2\beta_2^2 = \beta_1^2 + 2\beta_1\beta_2 + \beta_2^2} \quad \Rightarrow$$

$$\boldsymbol{\beta_{12}^2\left(1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2\right) = \beta_1^2 + 2\beta_1\beta_2 + \beta_2^2}$$

It is easy to see that the left and right sides of the equation contain squares of binomial formulas.

$$\boldsymbol{\beta_{12}^2(1 + \beta_1\beta_2)^2 = (\beta_1 + \beta_2)^2} \;\; \Rightarrow$$

It follows:

$$\boldsymbol{\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}}$$

Finally, replacing $\boldsymbol{\beta_x}$ by $\mathbf{v_x/c}$ yields the relativistic velocity addition formula:

$$\boldsymbol{v_{12} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}} \qquad (10.6)$$

The relation (10.6) agrees with the formula of Einstein's relativistic addition formula for velocities.

Of course, a derivation based on the law of energy conservation leads to the same result (see Appendix A IV).

It should be emphasized once again that the equation (10.6) has been derived only by the use of conservation laws. The postulate of the constancy of the speed of light has not been used against it.

Applying the law of energy and momentum conservation to the central collision of two particles makes it possible in the general case to derive the relativistic addition formula of velocities without using the postulate of the constancy of the speed of light.