12 Transformation of Space and Time Coordinate

The transformation for space and time can alternatively be derived from the proof of the length contraction (see chapter 11), without assuming the postulate of the constancy of the speed of light for any relative speeds between the light source and the receiver.

We consider two one-dimensional reference systems in motion relative to each other at constant speed. Two observers $\boldsymbol{O}$ and $\boldsymbol{O'}$ each rest on the coordinate origins $\boldsymbol{O}$ and $\boldsymbol{O'}$ of the two reference systems and measure the time $\boldsymbol{t}$ and $\boldsymbol{t'}$, respectively. The coordinate origins of the two reference systems coincide at time $\boldsymbol{t = 0}$ for $\boldsymbol{O}$ and $\boldsymbol{t' = 0}$ for $\boldsymbol{O'}$.

We can accept, as discussed in chapter 9, that for later times the time measurements $\boldsymbol{t}$ and $\boldsymbol{t'}$ may be different, i.e., we do not assume a priori that $\boldsymbol{t = t'}$, not even for $\boldsymbol{v \ll c}$.

To clearly illustrate the relationships between the reference systems, we will follow the following rules for the next diagrams:

- Each chart has an observer at rest. A second observer moves along the X axis at speed $\boldsymbol{v}$.
- The transformation is viewed from the point of view of the resting observer and thus his time is used. The observer at rest is highlighted in **gray** in the diagrams.
- The reference system of the moving observer is shown with **dashed lines**.
- The transformation always concerns the calculation of the space coordinate of the moving observer as a function of the space and time coordinates of the observer at rest.

<u>*Transformation of the space coordinates for $\boldsymbol{v \ll c}$*</u>

If the relative velocity $\boldsymbol{v}$ between the reference systems is considerably lower than the speed of light, the two observers do not perceive any length contractions in the direction of movement. In Figure 18, the observer $\boldsymbol{O}$ is at rest. $\boldsymbol{t}$ is the time measured by $\boldsymbol{O}$.

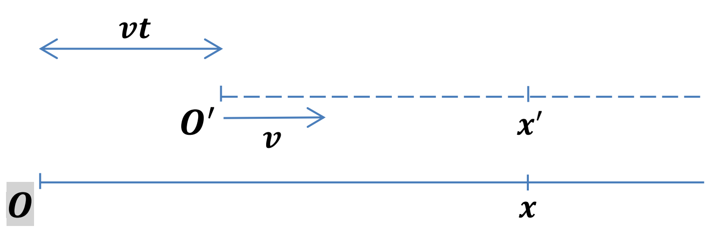

Fig. 18

According to Figure 18, from the point of view of $\boldsymbol{O}$, the relationships are:

$$\boldsymbol{x = x' + vt} \quad \Rightarrow \quad \boldsymbol{x' = x - vt} \qquad (12.1)$$

According to Figure 19 from the perspective of observer $\boldsymbol{O'}$ follows:

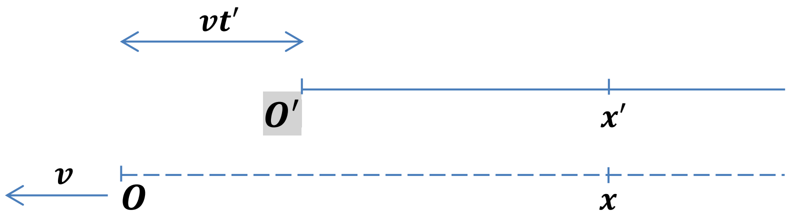

Fig. 19

The corresponding relations are:

$$x' = x - vt' \quad ➔ \quad x = x' + vt' \qquad (12.2)$$

And if one replaces $\boldsymbol{x'}$ by relation (12.1) then follows:

$$x = x - vt + vt' \quad ➔ \quad t = t' \qquad (12.3)$$

Relation (12.3) shows for $\boldsymbol{v} \ll \boldsymbol{c}$: The time coordinate is <u>invariant</u>.

The relations (12.1) and (12.3) correspond to the Galilean transformation and are valid in the context of classical mechanics for $\boldsymbol{v} \ll \boldsymbol{c}$:

$$x' = x - vt \quad (12.1); \qquad t = t' \quad (12.3)$$

<u>*Transformation of the space coordinates for any $\boldsymbol{v}$*</u>

Now consider the situation from the viewpoint of observer $\boldsymbol{O}$, this time for any velocity $\boldsymbol{v}$:

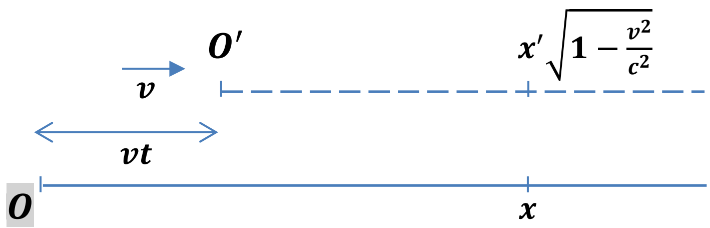

Fig. 20

As proved in chapter 11, observer $\boldsymbol{O}$ perceives a length contraction of the distance between $\boldsymbol{O'}$ and $\boldsymbol{x'}$, so that for him in the reference system $\boldsymbol{O'}$ the point corresponding to $\boldsymbol{x}$ is located at the coordinate $x'\sqrt{1-\frac{v^2}{c^2}}$, as shown in Figure 20.

From Figure 20 it can be seen that for observers $\boldsymbol{O}$ is valid:

$$x = x'\sqrt{1-\frac{v^2}{c^2}} + vt \qquad ==>$$

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (12.4)$$

$\boldsymbol{t}$ in (12.4) is the time from the point of view of $\boldsymbol{O}$.

Relation (12.4) represents the value of the space coordinate $\boldsymbol{x'}$ in the reference frame of the observer $\boldsymbol{O'}$, as a function of the space and time coordinate $(\boldsymbol{x}, \boldsymbol{t})$ of the reference system of $\boldsymbol{O}$.

Now let us look at the situation from the viewpoint of observer $\boldsymbol{O'}$:

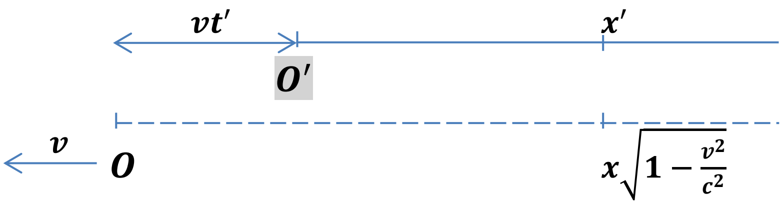

Fig. 21

Observer $\boldsymbol{O'}$ determines a length contraction of the distance between $\boldsymbol{O}$ and $\boldsymbol{x}$, so that for him the corresponding point of $\boldsymbol{x'}$ on the reference system $\boldsymbol{O}$ is found at the coordinate $\boldsymbol{x}\sqrt{\boldsymbol{1}-\frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}}$, as shown in Figure 21.

From Figure 21 it can also be deduced that for observers $\boldsymbol{O'}$ is valid:

$$\boldsymbol{x'} + \boldsymbol{vt'} = \boldsymbol{x}\sqrt{\boldsymbol{1}-\frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}} \qquad ==>$$

$$\boldsymbol{x} = \frac{\boldsymbol{x'} + \boldsymbol{vt'}}{\sqrt{\boldsymbol{1}-\frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}}} \qquad (12.5)$$

It should be noted that $\boldsymbol{t'}$ represents the time from the point of view of $\boldsymbol{O'}$. It is generally different from the time $\boldsymbol{t}$ of Observer $\boldsymbol{O}$, as discussed in chapter 9.

The relations (12.4) and (12.5) represent the spatial coordinate transformations for arbitrary velocities, from the perspective of two observers in relative motion to each other.

Transformation of the temporal coordinate:

The transformation for the temporal coordinate can be derived from the relations (12.4) and (12.5).

From relation (12.5) we get:

$$\boldsymbol{x'} = \boldsymbol{x}\sqrt{\boldsymbol{1}-\frac{\boldsymbol{v}^2}{\boldsymbol{c}^2}} - \boldsymbol{vt'} \qquad (12.6)$$

Relation (12.4) equated with relation (12.6) yields:

$$\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = x\sqrt{1 - \frac{v^2}{c^2}} - vt' \implies$$

$$x - vt = x\left(1 - \frac{v^2}{c^2}\right) - vt'\sqrt{1 - \frac{v^2}{c^2}} \implies$$

$$-vt = -x\frac{v^2}{c^2} - vt'\sqrt{1 - \frac{v^2}{c^2}} \implies$$

$$t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (12.7)$$

Using an analogous method, the following relation can be derived for t as a function of t' and x' :

$$t = \frac{t' + \frac{x'v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (12.8)$$

The coordinate transformations according to (12.4) and (12.5) as well as (12.7) and (12.8) appear in the same form as components of the Lorentz Transformation, which describes the transition between reference systems for arbitrary velocities in the framework of the special theory of relativity.

In this section it was shown how the space and time transformation for arbitrary relative velocities of observers can alternatively be derived from the phenomenon of relativistic length contraction.