3 Proofs of $E = mc^2$ from classical physics

About $E = mc^2$ Max Born wrote in his work "Die Relativitätstheorie Einsteins" (fifth edition, Springer-Verlag, page 244):

> *Einstein's equation $E = mc^2$, which determines the proportionality of energy and inertial mass, has often been called the most important result of the Relativity Theory. Therefore, let us give another simple proof that comes from Einstein himself and makes no use of the mathematical formalism of the Theory of Relativity [Emphasis added by the author]. This relies on the fact of the existence of the radiation pressure. That a wave of light which appears on an absorbing body exerts a pressure on it follows from Maxwell's field equations with the aid of a theorem first derived by Poynting (1884); that is, the momentum exerted by the energy E on the absorbing surface by a short flash or hit of light is equal to E/c. ... ".*

The derivation that follows this quote proves, contrary to the general belief, that the principle of equivalence of energy and mass is not a compelling result of the theory of relativity, because the derivation of the equation $E = mc^2$ cited by Max Born is based on a theorem from the year 1884. At that time there was only classical physics. Relativity Theory and Quantum Mechanics were developed later.

In the following I would like to present a similar proof of $\boldsymbol{E = mc^2}$ as the one mentioned above, which is based on the same phenomenon of the so-called "radiation pressure".

Derivation of $\boldsymbol{E = mc^2}$ based on the "radiation pressure".

The thought experiment considered here uses, instead of the effect of the radiation emitted in a tube, as described by Max Born, the observation of the phenomenon of emission and absorption of a photon between two physical bodies.

We consider a physical system consisting of two identical bodies K1 and K2 of mass $\boldsymbol{m}$, which initially rest at a distance $\boldsymbol{l}$ from each other.

It is assumed that the bodies do not exchange any energy or matter with the environment. It is also assumed that no external forces act on the bodies.

Because of these assumptions, the following conditions apply to any changes in the internal system state:

1. **The total mass of the system remains unchanged**
2. **The center of mass of the system remains at rest**

With the same masses, the center of gravity $\boldsymbol{B}$ of the system lies exactly midway between the bodies at a distance $\boldsymbol{l/2}$ to both, as shown in Figure 1.

Fig. 1 (see the animation)

It is assumed that at a certain point in time the left body K1 radiates a short intense beam of light towards the right body K2.

As a result of the emission of the light beam, the emitting body K1 receives a recoil and then moves in the opposite direction to the light beam at the speed $\boldsymbol{v}$.

We assume that after the time $\boldsymbol{\Delta t}$ from the emission the light beam has reached the body K2 and is absorbed by it. During the same time interval, the body K1 has covered the distance $\boldsymbol{\Delta l}$. Therefore, the following relation applies to the time interval ($\boldsymbol{c}$ = speed of light):

$$\Delta t = \frac{l}{c} = \frac{\Delta l}{v}$$

It follows:

$$v = \frac{\Delta l}{l} c \qquad (3.1)$$

Figure 2 shows the situation at the moment the light beam is absorbed by the right body.

According to the assumption, no external forces act on the system, so the position of body K2 has not changed in the meantime.

The body K1 on the left has moved away during the period $\boldsymbol{\Delta t}$ and is now $\boldsymbol{l + \Delta l}$ from the body K2.

After a cursory observation, one might think that the center of gravity $\boldsymbol{B}$ of the system would also have shifted. However, this is not possible because, according to the prerequisite, no external forces act on the system.

From this it can be concluded that because of the emission of the light beam, the body K1 on the left, which is located at the greater distance from the center of gravity, must have decreased by a certain mass $\boldsymbol{\Delta m}$ that has yet to be calculated. On the other hand, since the total mass of the system remains unchanged, the mass of the body K2 on the right must necessarily have increased by the same amount $\boldsymbol{\Delta m}$.

It follows that after the absorption of the light beam the mass of the body K2 has become $\boldsymbol{m + \Delta m}$ and the mass of the body K1 has become $\boldsymbol{m - \Delta m}$.

$m - \Delta m$ Fig. 2 $m + \Delta m$

The following applies to the position of the center of gravity between two masses as in Fig. 2:

$$(m - \Delta m)(\Delta l + l/2) = (m + \Delta m)\, l/2 \quad \Rightarrow$$

$$m\Delta l - \Delta m \Delta l + m\, l/2 - \Delta m\, l/2 = m\, l/2 + \Delta m\, l/2 \quad \Rightarrow$$

$$\frac{(m - \Delta m)\Delta l}{l} = \Delta m \qquad (3.2)$$

With the relation (3.2) we now have the second equation necessary to produce the proof.

The third equation required to derive the principle of equivalence is provided by Poynting's theorem from 1884, mentioned at the beginning of this chapter.

From this theorem it follows that for the momentum $\boldsymbol{p}$ of a light beam with energy $\boldsymbol{E}$: $\boldsymbol{p} = \boldsymbol{E}/\boldsymbol{c}$.

The law of conservation of momentum applied to the process of light emission shows that the counter impulse received from the left body K1 must be equal to the momentum of the light beam:

$$(\boldsymbol{m} - \boldsymbol{\Delta m})\boldsymbol{v} = \frac{\boldsymbol{E}}{\boldsymbol{c}} \qquad (3.3)$$

Using relation (3.1) in equation (3.3) gives:

$$(\boldsymbol{m} - \boldsymbol{\Delta m})\frac{\boldsymbol{\Delta l}}{\boldsymbol{l}}\boldsymbol{c} = \frac{\boldsymbol{E}}{\boldsymbol{c}} \quad \Rightarrow \quad (\boldsymbol{m} - \boldsymbol{\Delta m})\frac{\boldsymbol{\Delta l}}{\boldsymbol{l}} = \frac{\boldsymbol{E}}{\boldsymbol{c}^2} \qquad (3.4)$$

And taking into account relation (3.2) we then get:

$$\Delta \boldsymbol{m} = \frac{\boldsymbol{E}}{\boldsymbol{c}^2}$$

It can therefore be concluded that:

- The emission of a light beam with the energy $\boldsymbol{E}$ by an body causes a decrease of the mass of the body itself, which is equal to the energy of the light beam divided by the square of the speed of light.

- The absorption of a light beam with the energy $\boldsymbol{E}$ by an body causes an increase in the mass of the body itself, equal to the energy of the light beam divided by the square of the speed of light.

Derivation of $\boldsymbol{E} = \boldsymbol{mc}^2$ based on the Doppler effect

Another non-relativistic proof of the equivalence principle of energy and mass is based on the Doppler effect of electromagnetic radiation, as already mentioned in the introduction.

The Doppler Effect is treated in "classical" electrodynamics and does not represent a relativistic effect for low speeds of the emitting light source.

To begin with, we would like to refer to classical considerations:

The energy of a quantum of light, also called a photon, is equal to the product $\boldsymbol{hf}$, and the momentum of a photon is represented by the quotient $\boldsymbol{hf} \,/\, \boldsymbol{c}$ where $\boldsymbol{f}$ is the frequency assigned to the photon, $\boldsymbol{h}$ is Planck's constant and $\boldsymbol{c}$ is the speed of light in a vacuum.

It follows that between energy $\boldsymbol{E_f}$ and momentum $\boldsymbol{p_f}$ of a photon the following relationship exists: $\boldsymbol{E_f = p_f c}$.

We consider a body of mass $\boldsymbol{m_1}$ moving relative to an observer with a low velocity $\boldsymbol{v_1 \ll c}$.

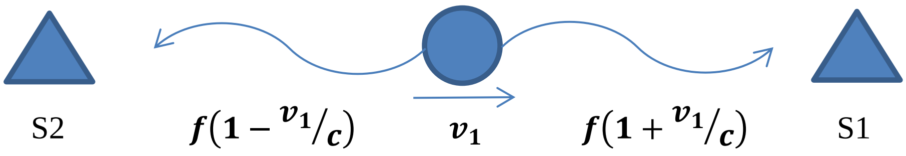

Fig. 3

We assume that at one point in time the body emits two photons of the same frequency: one photon in the direction of the movement, the other one in the opposite direction.

The energy emitted by the body is then: $\boldsymbol{E = 2hf}$.

According to Fig. 3, a spectrometer S1 will measure a frequency $\boldsymbol{f(1 + v_1/c)}$ for the photon in the direction of motion because of the optical Doppler Effect. A spectrometer S2 will measure a frequency $\boldsymbol{f(1 - v_1/c)}$ for the photon sent in the opposite direction.

According to the law of conservation of momentum, it follows that the momentum of the body before emission must be equal to the sum of the momenta of the body and the two photons after emission.

Therefore, we get from the point of view of the observer of the whole system[1]:

$$\boldsymbol{m_1 v_1 = m_2 v_2 + \frac{hf}{c}\left(1 + \frac{v_1}{c}\right) - \frac{hf}{c}\left(1 - \frac{v_1}{c}\right)} \qquad (3.5)$$

Here, the left-directed momentum is negative. It follows:

$$\boldsymbol{m_1 v_1 - m_2 v_2 = 2\frac{hf v_1}{c^2}} \qquad (3.6)$$

where $\boldsymbol{m_2}$ represents the mass and $\boldsymbol{v_2}$ represents the velocity of the body after emission.

[1] The derivation is made here with the use of the modern relations $f' = f(1 + v/c)$ and $f' = f(1 - v/c)$ for the optical Doppler effect. Instead, the relations of the Doppler-Fizeau effect from 1848 can also be used. At that time, Fizeau still distinguished between the movement of the light source and the receiver, based on the acoustic Doppler effect. He provided the expressions $f' = f/(1 - v/c)$ for the approach and $f' = f/(1 + v/c)$ for the removal of the source. Using these relations instead of the expressions mentioned above, equation (3.5) must be changed as follows:

$$m_1 v_1 = m_2 v_2 + \frac{hf}{c\left(1 - \frac{v_1}{c}\right)} - \frac{hf}{c\left(1 + \frac{v_1}{c}\right)}$$

Since this equation comes from classical physics, it can only be used for $v_1 \ll c$. The range of validity must therefore only be restricted to values of v_1, for which the result is that v_1^2/c^2 is negligibly small. It follows:

$$m_1 v_1 - m_2 v_2 = \frac{2hf v_1}{c^2\left(1 - \frac{v_1^2}{c^2}\right)}$$

Since in this last relation v_1^2/c^2 can be set to zero without any loss of value, it is reduced to relation (3.6). This proves that the derivation of the equivalence principle E-M can also be carried out with the relations for the Doppler effect by Fizeau from 1848.

Because of the symmetric emission (two equal photons in opposite directions) there will be no change in the velocity of the body after emission. Therefore $\boldsymbol{v_1} = \boldsymbol{v_2}$ is valid.

On the other hand, the mass of the body does not remain unchanged, otherwise would be $\boldsymbol{m_1 v_1} - \boldsymbol{m_2 v_2} = \boldsymbol{0}$ and, consequently, the term on the right side of (3.6) would be zero. But this could **only** be the case if, contrary to the assumption for the thought experiment, the frequency $\boldsymbol{f}$ or the velocity $\boldsymbol{v_1}$ would be zero.

Therefore, by substituting $\boldsymbol{v_2}$ for $\boldsymbol{v_1}$ and $\boldsymbol{m_1} - \boldsymbol{m_2}$ for mass decrease $\boldsymbol{\Delta m}$ in (3.6), the following equation is obtained.

$$\boldsymbol{\Delta m v_1} = \boldsymbol{2}\frac{\boldsymbol{hfv_1}}{\boldsymbol{c^2}}$$

After $\boldsymbol{v_1}$ is eliminated by division on both sides of this equation and considering that $\boldsymbol{2hf}$ is the energy radiated by the emission of the photons, the formula is given which describes the equivalence of mass and energy in the specific case of electromagnetic emission:

$$\boldsymbol{\Delta m} = \frac{\boldsymbol{E}}{\boldsymbol{c^2}} \quad \Leftrightarrow \quad \boldsymbol{E} = \boldsymbol{\Delta m c^2} \qquad (3.7)$$

That means:

The radiated energy of a body is equal to the product of its mass decrease and the square of the speed of light.

The alternative derivations of Einstein and the physicist Fritz Rohrlich are based on the principle of conservation of momentum and on the interaction between matter and electromagnetic radiation. Without making use of the mathematical formalism of relativity, they confirm the equivalence principle of energy and mass in the special case of electromagnetic emission.