14 The Lorentz Transformation and its application

With the coordinate transformations of Lorentz, which we derived in Chapter 12 on the basis of the law of conservation of energy and without the postulate of the constancy of the speed of light, we are now able to solve some important physical applications at high speeds and to discuss their results.

We consider an observer $\boldsymbol{O}$, which is at the origin of a one-dimensional frame of reference characterized by the coordinate $\boldsymbol{x}$. A second observer $\boldsymbol{O'}$ moves along the X-axis with constant velocity $\boldsymbol{v}$.

Assuming that for the time $\boldsymbol{t = 0}$ the positions of the observers $\boldsymbol{O}$ and $\boldsymbol{O'}$ match, the following transformations named after Lorentz apply:

$$\boldsymbol{x'} = \frac{\boldsymbol{x - vt}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}} \quad (14.1) \qquad and \qquad \boldsymbol{t'} = \frac{\boldsymbol{t - \frac{xv}{c^2}}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}} \quad (14.2)$$

In (14.1) and (14.2), $\boldsymbol{x'}$ and $\boldsymbol{t'}$ give the measurements for the position and the time of a point **P** from the point of view of the observer $\boldsymbol{O'}$. They are expressed as a function of the values of the position $\boldsymbol{x}$ and the time $\boldsymbol{t}$ measured by the observer $\boldsymbol{O}$ for the same point.

Solving (14.1) and (14.2) for $\boldsymbol{x}$ and $\boldsymbol{t}$ yields to:

$$\boldsymbol{x} = \frac{\boldsymbol{x' + vt'}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}} \quad (14.3) \qquad and \qquad \boldsymbol{t} = \frac{\boldsymbol{t' + \frac{x'v}{c^2}}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}} \quad (14.4)$$

Similarly, in (14.3) and (14.4), $\boldsymbol{x}$ and $\boldsymbol{t}$ give the measurements for the position and time of a point **P** as seen by the observer $\boldsymbol{O}$. They are determined by the position $\boldsymbol{x'}$ and the time $\boldsymbol{t'}$ expressed by the observer $\boldsymbol{O'}$ for the same point.

It is important to note that the point **P** does not have to rest in one of the two reference systems. Consequently, both $\boldsymbol{x}$ and $\boldsymbol{x'}$ can be time dependent. Since only uniform motions are considered in this essay, we consider only the following temporal dependencies of $\boldsymbol{x}$ and $\boldsymbol{x'}$:

$$\boldsymbol{x = x_0 + v_p t} \quad (14.5) \qquad and \qquad \boldsymbol{x' = x'_0 + v'_p t'} \quad (14.6)$$

With the parameters:

$\boldsymbol{v_p}$ is the velocity of **P** in $\boldsymbol{O}$.

$\boldsymbol{x_0}$ is the position at time point $\boldsymbol{t = 0}$ of **P** in $\boldsymbol{O}$.

$\boldsymbol{v'_p}$ is the velocity of **P** in $\boldsymbol{O'}$.

$\boldsymbol{x'_0}$ is the position at time point $\boldsymbol{t' = 0}$ of **P** in $\boldsymbol{O'}$.

From (14.1) and (14.3) it follows that if $\boldsymbol{x_0 = 0}$ then $\boldsymbol{x'_0 = 0}$ and vice versa.

Also, if $\boldsymbol{v'_p = 0}$, then $\boldsymbol{v_p = v}$ and if $\boldsymbol{v_p = 0}$, then $\boldsymbol{v'_p = -v}$ and vice versa.

In summary:

$$\boldsymbol{x_0 = 0} \quad <==> \quad \boldsymbol{x'_0 = 0} \qquad (14.7)$$

$$\boldsymbol{v'_p = 0} \quad ==> \quad \boldsymbol{v_p = v} \qquad (14.8)$$

$$\boldsymbol{v_p = 0} \quad ==> \quad \boldsymbol{v'_p = -v} \qquad (14.9)$$

The following are application examples of the Lorentz Transformation

1) <u>Time of a point at rest in frame $\boldsymbol{O}$ - Time is not absolute</u>

A point **P** rests in the reference system $\boldsymbol{O}$ ($\boldsymbol{v_p = 0}$) at the coordinates $\boldsymbol{x_0 = 0}$ and $\boldsymbol{t = 0}$.

We get from relation (14.2):

$$\boldsymbol{t'} = \frac{-\dfrac{\boldsymbol{x_0 v}}{\boldsymbol{c^2}}}{\sqrt{\boldsymbol{1 - \dfrac{v^2}{c^2}}}}$$

For $\boldsymbol{x_0 > 0}$ and $\boldsymbol{v > 0}$ applies: $\boldsymbol{t' < 0}$.

So, while observer $\boldsymbol{O}$ measures time $\boldsymbol{t = 0}$ for point $\boldsymbol{x_0}$, observer $\boldsymbol{O'}$ sees for the same point a time already in the past.

2) <u>Length of a bar at rest in the reference frame $\boldsymbol{O'}$- Length contraction</u>

A bar is at rest in the frame of reference of the observer $\boldsymbol{O'}$ and has the length $\boldsymbol{l'}$. The bar ends have the coordinates $\boldsymbol{x_1 = 0}$ and $\boldsymbol{x_2 = l}$ at the time of the observer $\boldsymbol{O}$ at $\boldsymbol{t = 0}$.

From relation (14.1) results for the bar ends in reference system $\boldsymbol{O'}$:

$$\boldsymbol{x'_1 = 0}\,;\; \boldsymbol{x'_2} = \frac{\boldsymbol{l}}{\sqrt{\boldsymbol{1 - \dfrac{v^2}{c^2}}}} \qquad ==>$$

$$\boldsymbol{x'_2 - x'_1} = \boldsymbol{l'} = \frac{\boldsymbol{l}}{\sqrt{\boldsymbol{1 - \dfrac{v^2}{c^2}}}} \qquad ==>$$

For the bar length in frame $\boldsymbol{O}$ the following applies:

$$l = l'\sqrt{1-\frac{v^2}{c^2}}$$

Observer O perceives a length contraction for the bar resting in the reference system of the observer $\underline{O'}$.

3) Proper time t' of the observer O' - Time dilation

For the observer O' applies in frame O: $x_0 = 0$ and $v_p = v$. From (14.5) follows: $x = vt$. Inserted in (14.2) results in:

$$t' = \frac{t - \frac{tv^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad ==> \quad t' = t\sqrt{1-\frac{v^2}{c^2}} \quad ==> \quad t = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

O perceives a time dilation with respect to O'. In other words, the time intervals that pass in system O' appear extended in system O.

4) Photon starts in the reference system O' at $t' = 0$ *and at* $x'_0 = 0$ - Constancy of the speed of light

With $x'_0 = 0$ and $v'_p = c$ follows from (14.6): $x' = ct'$. This inserted in (14.3) and (14.4) leads to:

$$x = \frac{ct' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad and \qquad t = \frac{t' + \frac{ct'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

As for $x'_0 = 0$ also $x_0 = 0$, it follows from (14.5) for the velocity of the photon in the reference system O:

$$v_p = \frac{x}{t} \quad ==> \quad v_p = \frac{t'(c+v)}{t'(1+\frac{v}{c})} \quad ==> \quad v_p = c$$

Despite the relative velocity v between the reference systems, both observer O and O' will measure from their perspective that the photon moves at the same velocity c.

5) Moving point in the reference frame O' - Speed addition

A point has the position $x' = 0$ for $t' = 0$ in the frame O'. From (14.6) follows: $x'_0 = 0$ and consequently $x_0 = 0$. The relations (14.5) and (14.6) are reduced to:

$$x = v_p t \qquad and \qquad x' = v'_p t'$$

These are now used in (14.3) and (14.4):

$$v_p t = \frac{v'_p t' + v t'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (a) \qquad and \qquad t = \frac{t' + \frac{v'_p v t'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (b)$$

From (b) results:

$$t' = \frac{t\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v'_p v}{c^2}}$$

And inserted in (a) we get:

$$v_p = \frac{v'_p + v}{1 + \frac{v'_p v}{c^2}} \qquad (c)$$

Equation (c) stands for the relativistic speed addition.

6) Two photons reach observer O' at the same time. Is this also the case from the point of view of O?- Simultaneity

Two photons start in the reference system O' at the time $t' = 0$ from the positions $-x'_0$ and x'_0 in the direction O'. From (14.6) applies to the first and the second photon:

$$x'_1 = -x'_0 + ct' \quad und \quad x'_2 = x'_0 - ct'$$

The two photons simultaneously reach the position of the observer O' at $x' = 0$ after the time $t' = {x'_0}/{c}$.

The starting time of the photons can be calculated from observer O view by relation (14.4).

At the starting point for the first photon applies: $t' = 0$ and $x' = -x'_0$.

From (14.4) follows:

$$t_{01} = \frac{\frac{-x'_0 v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

At the starting point for the second photon applies: $t' = 0$ and $x' = x'_0$.

From (14.4) follows:

$$\boldsymbol{t_{02}} = \frac{\frac{\boldsymbol{x'_0 v}}{\boldsymbol{c^2}}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}}$$

One sees: for observers $\boldsymbol{O}$, the photons do not start simultaneously. The first photon starts earlier.

At what point in time do the photons reach the position of the observer $\boldsymbol{O}$ from the point of view of the observer $\boldsymbol{O'}$?

We have seen that the following applies to the first and the second photon:

$$\boldsymbol{x'_1 = -x'_0 + ct'} \quad \textit{and} \quad \boldsymbol{x'_2 = x'_0 - ct'}$$

Inserted in (14.4), we get for the times of the two photons:

$$\boldsymbol{t_1} = \frac{\boldsymbol{t'} + \frac{\boldsymbol{(-x'_0 + ct')v}}{\boldsymbol{c^2}}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}} \quad \textit{and} \quad \boldsymbol{t_2} = \frac{\boldsymbol{t'} + \frac{\boldsymbol{(x'_0 - ct')v}}{\boldsymbol{c^2}}}{\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}}$$

Since it is the time $\boldsymbol{t'} = \boldsymbol{x'_0}/\boldsymbol{c}$ that passes for both photons in frame $\boldsymbol{O'}$, we get:

$$\boldsymbol{t_1 = t_2} = \frac{\boldsymbol{x'_0}}{\boldsymbol{c}\sqrt{\boldsymbol{1 - \frac{v^2}{c^2}}}} = \frac{\boldsymbol{x'_0}}{\sqrt{\boldsymbol{c^2 - v^2}}}$$

The photons thus reach the position of observer $\boldsymbol{O'}$ simultaneously for observer $\boldsymbol{O}$ as well.

For observer $\boldsymbol{O}$ the observer $\boldsymbol{O'}$ moves at velocity $\boldsymbol{v}$. Therefore, the first photon must travel a longer distance up to the observer $\boldsymbol{O'}$ than the second at the same speed $\boldsymbol{c}$. Nevertheless, the photons reach the observer $\boldsymbol{O'}$ at the same time, because from the observer's $\boldsymbol{O}$ point of view, the first photon starts earlier than the second.