11 Derivation of length contraction and time dilation

In the next thought experiment, we will see how the length contraction can be derived as a function of the velocity using the law of conservation of energy.

For this purpose, we imagine the central collision between two electrons which have a distance $2\boldsymbol{l}$ to each other at time $\boldsymbol{t} = \boldsymbol{0}$ and which move towards each other with equal velocities $\boldsymbol{v}$, as illustrated on the left in Fig. 15.

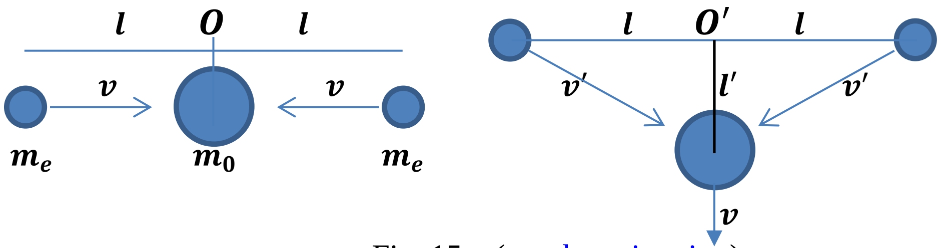

Fig. 15 (see the animation)

Let us assume that as a result of the collision a new particle forms with the mass $\boldsymbol{m_0}$. From the point of view of an observer $\boldsymbol{O}$, this particle is in the origin of a reference system at rest to him.

Since for observer $\boldsymbol{O}$ the length of the distance traveled by a particle to the collision is equal to $\boldsymbol{l}$, depending on the time $\boldsymbol{t}$ until the collision $\boldsymbol{l} = \boldsymbol{vt}$ is valid.

Let us now look at the same thought experiment from the point of view of a second observer $\boldsymbol{O'}$, who rests in a reference system that moves at the same velocity $\boldsymbol{v}$ as the particles, but vertically upwards (see Figure 15, right).

Since this motion in y-direction is orthogonal to the direction of motion of the colliding electrons, the observers $\boldsymbol{O}$ and $\boldsymbol{O'}$ in x-direction measure the same length $\boldsymbol{l}$, the same velocity $\boldsymbol{v}$ and thus, until the collision, also the same time $\boldsymbol{t}$.[1]

During this time, the observer moves forward in the coordinate system $\boldsymbol{O'}$ by the distance $\boldsymbol{l'}$, i.e., the collision takes place for him on the y-axis shifted backwards by $\boldsymbol{l'}$ (see Figure 15 right).

We assume that at time $\boldsymbol{t} = \boldsymbol{0}$ the two origins of the coordinate systems $\boldsymbol{O}$ and $\boldsymbol{O'}$coincide and that for the observer $\boldsymbol{O'}$ the electrons collide on the Y-axis at the point T at the coordinate $\boldsymbol{l'}$ as shown in Figure 16.

[1] In general, every observer needs transformations to express space and time from the perspective of another observer. These transformations are still unknown in the derivation of the length contraction. Nevertheless, it is certain that for a relative speed $\boldsymbol{v} = \boldsymbol{0}$ the observers measure the same length and time. This is a two-dimensional case. So that we can orientate ourselves better, I call X the axis on which the electrons move and Y the axis on which observer $\boldsymbol{O'}$ moves. In two dimensions, each of the two observers needs a transformation for the X-axis and a transformation for the Y-axis in order to calculate the lengths from the perspective of the other observer. Both transformations are only dependent on the component $\boldsymbol{v_x}$ and $\boldsymbol{v_y}$ of the relative speed $\boldsymbol{v}$ of the observers. Here only the component of the movement of the particles in the X-direction is needed for both observers to calculate $\boldsymbol{l}, \boldsymbol{v}$ and $\boldsymbol{t}$. The transformation for the X-axis is sufficient for this. The component $\boldsymbol{v_x}$ of the relative speed $\boldsymbol{v}$ between the observers in the X direction is, however, zero. This is why the transformation for the X-axis provides the same values for lengths and times for both observers.

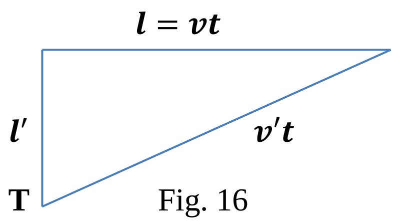

Fig. 16

From Fig. 16 results[2]:

$$v'^2 = v^2 + \frac{l'^2}{t^2} \qquad (11.1)$$

In equation (11.1) there are two unknown variables: the length l' and the velocity v'.

To calculate the length l' we therefore need a second relation that expresses the velocity v' in the oblique direction only as a function of the velocity v.

We will calculate l' at first for a speed v much lower than the speed of light.

Calculation of l' for $v \ll c$

As can be seen from Figure 17, the velocity v' of the electrons measured by observer O' for $v \ll c$ results from the sum of two mutually orthogonal vector components of the same magnitude.

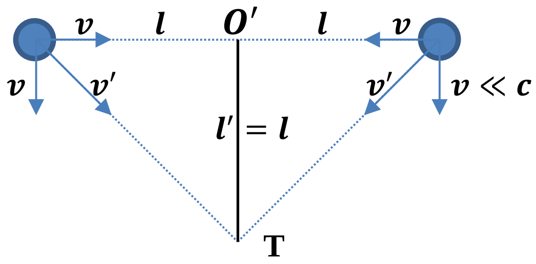

Fig. 17

The vector v' thus has an inclination of 45°, therefore for $v \ll c$ the electrons meet from the point of view of O' at the point **T** at

$$l' = l$$

Under these conditions, the vector v' has the value $\sqrt{2}v$, which, however, would assume an impermissible value greater than c from approximately $v \cong 70\%\ c$.

Thus, the boundary of this classical view for $v \ll c$ becomes clear.

[2] The time interval t measured by the observers on the X-axis until the collision is necessarily the same as that measured by observer O' on the hypotenuse of the right triangle in Fig. 16. If it were not so, instead of one, observer O' would experience two collisions at different times, the first on the inclined path and the second as a mapping of the first on the X-axis.

Calculation of $\boldsymbol{l'}$ for any velocity:

To calculate the distance $\boldsymbol{l'}$ for arbitrary velocities, we need another equation for the unknown velocity $\boldsymbol{v'}$.

Here it helps us that when calculating $\boldsymbol{v'}$ the conservation of the momentum and the energy must be considered.

This requirement makes it possible to calculate the length $\boldsymbol{l'}$ from a relativistic point of view as follows:

From $\boldsymbol{O}$ point of view, the following applies because of the law of conservation of energy (see Figure 15 left):

$$\boldsymbol{m_0c^2} = \frac{\boldsymbol{2m_{0e}c^2}}{\sqrt{\boldsymbol{1-\frac{v^2}{c^2}}}} \qquad (11.2)$$

From observer's $\boldsymbol{O'}$ point of view, the formed particle moves further down along the Y-axis after the collision with velocity $\boldsymbol{v}$ (see right in Fig. 15). For $\boldsymbol{O'}$ the result is (see also Appendix A V):

$$\frac{\boldsymbol{m_0c^2}}{\sqrt{\boldsymbol{1-\frac{v^2}{c^2}}}} = \frac{\boldsymbol{2m_{0e}c^2}}{\sqrt{\boldsymbol{1-\frac{v'^2}{c^2}}}} \qquad (11.3)$$

By using (11.2) in (11.3) we get:

$$\frac{\boldsymbol{2m_{0e}c^2}}{\boldsymbol{1-\frac{v^2}{c^2}}} = \frac{\boldsymbol{2m_{0e}c^2}}{\sqrt{\boldsymbol{1-\frac{v'^2}{c^2}}}} \quad ==>$$

$$\boldsymbol{1-\frac{v^2}{c^2}} = \sqrt{\boldsymbol{1-\frac{v'^2}{c^2}}} \quad ==>$$

$$\boldsymbol{1-2\frac{v^2}{c^2}+\frac{v^4}{c^4}} = \boldsymbol{1-\frac{v'^2}{c^2}} \quad ==>$$

$$\boldsymbol{v'^2} = \boldsymbol{2v^2-\frac{v^4}{c^2}} \qquad (11.4)$$

Equation (11.4) is the searched relation, which expresses $\boldsymbol{v'}$ in dependence of $\boldsymbol{v}$ and because of equation (11.1) now follows:

$$\frac{\boldsymbol{l'^2}}{\boldsymbol{t^2}} = \boldsymbol{v^2}\left(\boldsymbol{1-\frac{v^2}{c^2}}\right) \qquad (11.5)$$

Since $\boldsymbol{vt = l}$, (11.5) can be rewritten as follows:

$$\boldsymbol{l'^2 = l^2\left(1 - \frac{v^2}{c^2}\right)}$$

And it follows:

$$\boldsymbol{l' = l\sqrt{1 - \frac{v^2}{c^2}}} \qquad (11.6)$$

Relation (11.6), in accordance with the theory of relativity, expresses the velocity-dependent length contraction in the direction of motion.

The time dilation can also be derived in a simple manner from relation (11.6).

The following applies to the vertical direction:

$$\boldsymbol{t' = \frac{l'}{v}} \quad \Rightarrow \quad \boldsymbol{t' = \frac{l\sqrt{1 - \frac{v^2}{c^2}}}{v}} \quad \Rightarrow \quad \boldsymbol{t' = t\sqrt{1 - \frac{v^2}{c^2}}} \qquad (11.7)$$

The relation (11.7) expresses, in accordance with the theory of relativity, the speed-dependent time dilation.

> The application of the law of conservation of energy to the collision of two electrons from the point of view of observers in relative motion to each other makes it possible to derive the relation of relativistic length contraction and time dilation depending on velocity.