5 Dependence of Mass on Velocity[1]

In this chapter we will see how the formula for the dependence of inertial mass on velocity can be derived from the second law of dynamics in conjunction with the Mass–Energy Equivalence Principle.

We assume that a constant force $\boldsymbol{F}$ acts on a point mass.

In the first chapter it was shown that if the path runs in the same direction of the force, the infinitesimal work of the force can be expressed by the following differential equation:

$$\boldsymbol{Fds = dE = mvdv + v^2dm} \qquad (1.5)$$

In (1.5), $\boldsymbol{m}$ is a measure of the inertia of the physical body. From a physical point of view, however, it is not just the mass of the body, but the mass of the entire "system" that consists of the body with its kinetic energy, because during this section we will see that not only does the body itself have a mass, but also its energy.

The relation (1.5) shows that an energy input generally not only causes an increase in the velocity of the point mass ($\boldsymbol{mvdv}$), but also causes an increment of its inertia ($\boldsymbol{v^2dm}$).

If the speed is considerably lower than the speed of light, then only the contribution of the mass of the body is significant for the inertia since the kinetic energy remains low.

At high speeds, the mass of kinetic energy is no longer negligible. It becomes prevalent at speeds close to that of light, thus inhibiting the acceleration of the particles.

If it is assumed that a constant electrical force acts on a progressively increasing mass, then it should be understandable why a particle can only be accelerated to a certain speed and not further.

It should be noted that this last point is particularly important for an intuitive understanding of the process that will lead to the derivation of the first relativistic equation in this chapter.

In the first chapter it was mentioned that the equation (1.5) cannot be solved by integration, unless one knows another relationship between energy and mass.

The derivations made in the previous chapters 3 and 4 fill this gap by providing the missing relation, because it can be concluded from the equation of the equivalence of mass and energy $\boldsymbol{E = \Delta mc^2}$ that not only mass can transform into energy, but also that every energy supply is accompanied by an increase in mass.

In other words, *"mass is energy, and energy has mass."*[2]

Based on this, it can be stated that the inertia, which can be assigned to the supplied energy $\boldsymbol{dE}$ in (1.5), corresponds to the following mass increase $\boldsymbol{dm}$:

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[1] The following derivation has been prepared by me independently of other physicists in November 2016. It was only after the publication of the first edition of this work that I learned from a reader that a similar derivation was already made in 1961 by Professor Franz von Krbek in his book "Fundamentals of Mechanics".

[2] Albert Einstein, Leopold Infeld – Die Evolution der Physik, page 267 – Weltbild Verlag

$$\boldsymbol{Fds = dE = c^2dm} \qquad (5.1)$$

The substitution of $\boldsymbol{Fds}$ by $\boldsymbol{c^2dm}$ makes it possible to eliminate the differential $\boldsymbol{ds}$ from equation (1.5). Thus, an integrable differential equation results only as a function of mass and velocity:

$$\boldsymbol{c^2dm = mvdv + v^2dm} \qquad (5.2)$$

The result of the integration of (5.2) gives the relation of the dependence of the mass on the velocity.

The relation (5.2) can also be rewritten as follows:

$$\boldsymbol{\frac{dm}{m} = \frac{v}{c^2 - v^2}dv} \qquad (5.3)$$

If the second side of equation (5.3) is integrated between the integration limits **0** and the indefinite value $\boldsymbol{v}$ of the velocity and $\boldsymbol{m_0}$ denotes the mass corresponding to zero velocity (the so-called rest mass), then:

$$\boldsymbol{\int_{m_0}^{m}\frac{dm}{m} = \int_0^v \frac{v}{c^2 - v^2}dv = -\frac{1}{2}\int_0^v \frac{d(c^2 - v^2)}{c^2 - v^2}} \qquad \Rightarrow$$

$$\boldsymbol{[ln(m)]_{m_0}^{m} = -\frac{1}{2}[\ln(c^2 - v^2)]_0^v} \qquad \Rightarrow$$

$$\boldsymbol{ln\frac{m}{m_0} = \frac{1}{2}ln\frac{c^2}{c^2 - v^2}} \qquad \Rightarrow$$

$$\boldsymbol{\frac{m}{m_0} = \sqrt{\frac{c^2}{c^2 - v^2}}} \qquad \Rightarrow$$

$$\boldsymbol{m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad (5.4)$$

The relation (5.4) expresses the dependence of the inertia of a body of mass $\boldsymbol{m_0}$ on the velocity.

Thus, one of the most important results of the Special Theory of Relativity is confirmed without the use of the Lorentz transformations, which are the basis of Einstein's theory.

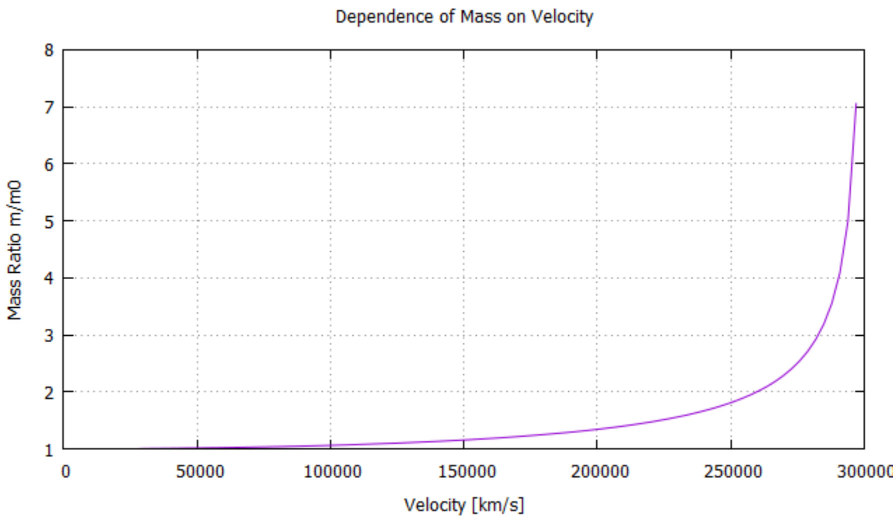

Fig. 5

In his work "Theory of Relativity" Wolfgang Pauli explains:

> *"This expression of mass dependence on velocity was first derived by Lorentz for the mass of the electron, assuming that also the electrons undergo the Lorentz contraction during motion."*[3]

One thing is certain: With the alternative derivation of the dependence of mass on velocity, we have finally left the field of application of classical physics and entered the field of relativity theory, especially since the formula in question contains the Lorentz factor.

It is noteworthy, however, that in the present work the formula (5.4) was derived starting from classical physics and without the hypothesis of length contraction, as is the case with relativistic proof based on the Lorentz transformation.

In the further course of this work, the relationship (5.4) is often used to derive further relativistic formulae.

Thus, $\boldsymbol{m_0}$ always means the invariant mass (also called rest mass) of a body at zero velocity. Instead, $\boldsymbol{m}$ denotes the total mass that can be assigned to a body as a function of its speed[4].

Starting from (5.4), when both sides of the equation are multiplied by velocity, the formula for the momentum in the general case is:

$$\vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (5.5)$$

From the relation (5.5) it can be seen, in agreement with the experimental observations, that a body can neither exceed nor reach the speed of light.

[3] Wolfgang Pauli – Relativitätstheorie, page 97 – Springer Verlag
[4] It should be clarified at this point: To the inertia of the body, two mass fractions contribute (i) on the one hand, the mass $\boldsymbol{m_0}$ of the body itself, and (ii) on the other hand, the mass that can be assigned to the kinetic energy of the body. The last part can be several orders of magnitude larger than the mass $\boldsymbol{m_0}$ of the body.

The immediate consequence of this conclusion is the inapplicability of the Galilean transformation at any speed, with the consequence that the help of transformations must be dispensed with in the further course of this treatise.

At this point we would like to anticipate that multiplying both sides of equation (5.4) by the square of the speed of light gives the formula for the total (i.e., internal + kinetic) energy of a body as a function of its velocity:

$$\boldsymbol{mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad (5.6)$$

The proof will be provided in the next chapter.

The proof of the principle of equivalence between energy and mass given in chapters 3 and 4 makes it possible to assign its corresponding inertia to the energy transferred to a body. Thus, one obtains the necessary relation to the solution of differential equilibrium from the second law of dynamics. By integration one obtains the relation of the dependence of the inertia of a body on its speed. In this way, one of the most important results of the special theory of relativity is proved without resorting to the Lorentz transformation and thus without presupposing the length contraction.