6 Calculation of Kinetic and Total Energy

With the help of the definition of work in mechanics, the kinetic energy can be calculated, which is supplied to an unbound rigid body by a force acting on it.

Within the framework of classical mechanics, one usually proceeds as follows:

Assuming the mass is constant, the differential equation (1.2) can be used.

As seen in the first chapter, (1.2) is a direct consequence of equation (1.1):

$$\vec{F} = m_0\vec{a} \quad \Leftrightarrow \quad \vec{F} = m_0\frac{d\vec{v}}{dt} \qquad (1.1)$$

Consequently, for the supply of kinetic energy by the infinitesimal element of the work Fds is valid[1]:

$$dE_k = Fds = m_0 v dv \qquad (1.2)$$

If the differential equation (1.2) is integrated for an initial velocity equal to zero, the expression of the kinetic energy results:

$$E_k = m_0 \int_0^v v\, dv = \frac{1}{2}m_0 v^2 \qquad (6.1)$$

Since it was derived from equation (1.1), the relation (6.1) describes the kinetic energy of a point mass m_0 only for speeds significantly lower than that of light, where the body's inertia remains virtually unchanged.

In the more general case, i.e., even for speeds approaching the speed of light, it is necessary to use the following relation instead ...

$$dE_k = Fds = v^2 dm + mvdv \qquad (1.5)$$

... in accordance with what was stated in the first chapter[2].

If the two relationships considered in the fifth chapter are applied for the mass, ...

$$Fds = dE_k = c^2 dm \qquad (5.1)$$

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (5.4)$$

... then it can be stated that by substitution, the velocity-dependent mass m can be eliminated from the differential equation (1.5).

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[1] As already mentioned, it is also assumed here that the infinitesimal distance element ds runs in the same direction of the force F.

[2] It should be remembered that at this point, as in the remainder of this work, always unbound rigid bodies are considered that have no potential energy. The bodies considered here can therefore be adapted to the subatomic particles. It is therefore obvious that the infinitesimal work in the differential equation (1.5) completely merges into the kinetic energy of the point mass.

This results in the required relationship to solve the differential equation (1.5).

From (5.1) follows:

$$dm = \frac{dE_k}{c^2} \qquad (6.2)$$

After the substitution of (5.4) and (6.2) in (1.5) the following relation between supplied mechanical energy and velocity results:

$$dE_k = v^2 \frac{dE_k}{c^2} + \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} v dv \qquad \Rightarrow$$

$$(1 - \frac{v^2}{c^2}) dE_k = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} dv \qquad \Rightarrow$$

$$dE_k = \frac{m_0 v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv \qquad (6.3)$$

If the equations (1.2) and (6.3) are compared with each other, it can be stated that in both relations the differential of the kinetic energy dE_k is given only as a function of the invariant mass m_0 and the velocity v.

However, the relation (6.3) is the general expression for the calculation of the kinetic energy at arbitrary speeds.

It is easy to show that (6.3) can be reduced to (1.2) for $v << c$.

Just as equation (1.2) provides by integration the kinetic energy of a constant point mass m_0 at low speeds, the integration of the differential equation (6.3) leads to the calculation of the kinetic energy for the general application at arbitrary speeds.

To calculate the kinetic energy, the differential equation (6.3) is integrated between the integration limits 0 and v analogously to (1.2):

$$E_k = m_0 \int_0^v \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} v dv \qquad \Rightarrow$$

$$E_k = -\frac{1}{2} m_0 c^2 \int_0^v \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} d\left(1 - \frac{v^2}{c^2}\right) \qquad \Rightarrow$$

$$E_k = -\frac{1}{2}m_0c^2\left[\frac{\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_0^v \qquad \Rightarrow$$

$$E_k = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right) \qquad \Rightarrow$$

$$E_k = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2 \qquad (6.4)$$

The relation (6.4) represents the expression of the kinetic energy of a body as a function of its mass and velocity.

It agrees with the relativistic formula of kinetic energy.

By means of the series expansion of Taylor and also by the following algebraic method it can be shown that the relation (6.4) for $v << c$ reduces to the equation (6.1):

For velocities significantly lower than that of light, the quotient $v^4/4c^4$ is negligible compared to the term v^2/c^2. Therefore, this quotient can be added to the radical of equation (6.4) with no value change:

$$E_k = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}+\frac{v^4}{4c^4}}} - m_0c^2 \qquad \Rightarrow$$

Since the denominator is now the root of the square of a binomial formula, we get:

$$E_k = \frac{m_0c^2}{1-\frac{v^2}{2c^2}} - m_0c^2 \qquad \Rightarrow$$

$$E_k = \frac{\cancel{m_0c^2} - \cancel{m_0c^2} + \frac{1}{2}m_0v^2}{1-\frac{v^2}{2c^2}}$$

This equation then reduces to (6.1) for $v << c$.

Fig. 6 compares the kinetic energy curves from equations (6.1) (green curve) and (6.4) (purple curve).

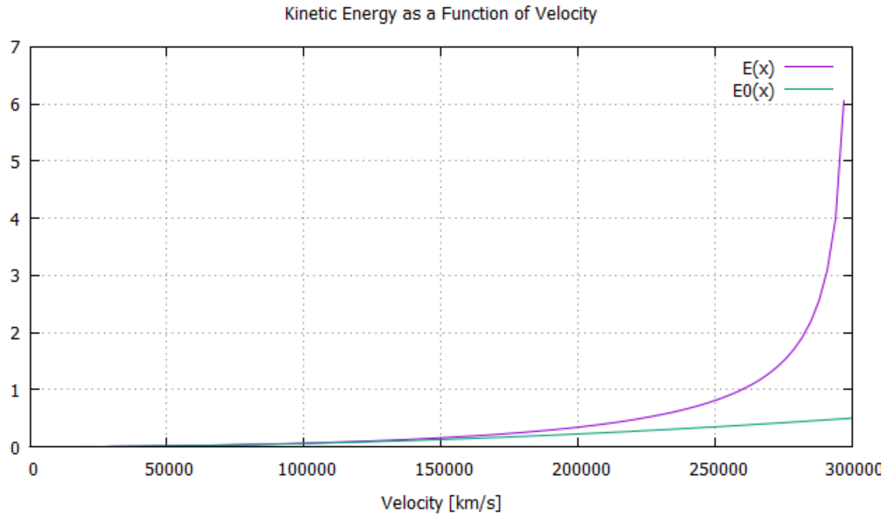

Fig. 6

It can be seen that for low speeds the two curves actually coincide.

However, the curves diverge more and more as the velocities approach the speed of light.

Considering the relation (5.4), the formula (6.4) can also be written as follows:

$$\boldsymbol{mc^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = E_k + m_0 c^2} \qquad (6.5)$$

By the relation (6.5) we come to the following important result:

Since the right side of the formula (6.5) is equal to the sum of kinetic and internal energy, it can be concluded that $\boldsymbol{mc^2}$ or $\boldsymbol{m_0 c^2 / \sqrt{1-\frac{v^2}{c^2}}}$ represents the total energy of the point mass $\boldsymbol{m_0}$ as a function of the velocity.

This confirms what was anticipated in the previous section by Equation (5.6) but has not yet been proven.

> The relations between the work and the mass as a function of the velocity derived in chapters 5 and 6 allow substitution to eliminate the velocity-dependent mass from the differential equation of the work. The subsequent integration then yields as an end result the expression of the kinetic energy and, at the same time, of the total energy of a point mass as a function of its velocity. These equations also agree with the relativistic formulas.