16 Dependence of Acceleration on Velocity

It has already been pointed out in chapter 2 that the second law of dynamics in connection with the mass formula (5.4) leads to the relation of relativistic acceleration (see Appendix A II).

To calculate the acceleration, the second law of dynamics is also used in this section, which, as stated in the first chapter, can be expressed in the more general case by the following relation:

$$\vec{F} = \frac{d(m\vec{v})}{dt} \Leftrightarrow \vec{F} = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt} \qquad (16.1)$$

To derive the speed dependence of the acceleration, two proofs are presented below:

In the first proof, for the sake of simplicity, the case is considered in which a physical body moves in the same direction as the force acting on it.

In this first case, a derivation is presented which is based on a purely scalar calculation and which is suitable for describing the movement of particles in linear accelerators.

In the second case, a vector calculation is used to derive the two vector components (longitudinal and transverse components) of the acceleration. This second proof covers the situations in which a physical body does not move in the same direction as the force acting on it, as e.g. occurs in the movement of celestial bodies.

Both cases are suitable to show that the second principle of dynamics represents the fundamental relation for calculating the relativistic acceleration.

Case 1. Scalar calculation.

In this derivation, too, we will limit ourselves to linear motions in which, as is the case in linear accelerators, the path of the particles runs in the same direction of the force.

In this case, relation (16.1) can be used in scalar form:

$$F = v\frac{dm}{dt} + m\frac{dv}{dt} \qquad (16.2)$$

Since the path is in the same direction of force, relation (5.1) (see chapter 5) can be used for infinitesimal work:

$$Fds = c^2 dm \qquad (5.1)$$

From (5.1) follows:

$$dm = \frac{Fvdt}{c^2} \qquad (16.3)$$

After substituting (16.3) and the relativistic relation $m = m_0/\sqrt{1 - v^2/c^2}$ into (16.2) and reducing variables we obtain:

$$F = \frac{v^2}{c^2}F + \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{dv}{dt}$$

It follows:

$$F = \frac{m_0 a}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \qquad (16.4)$$

From equation (16.4) the formula of linear acceleration as a function of velocity can be derived:

$$a = \frac{F}{m_0}\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}} \qquad (16.5)$$

It is easy to see that (16.5) can be simplified to scalar form of relation (1.1) if $v \ll c$.

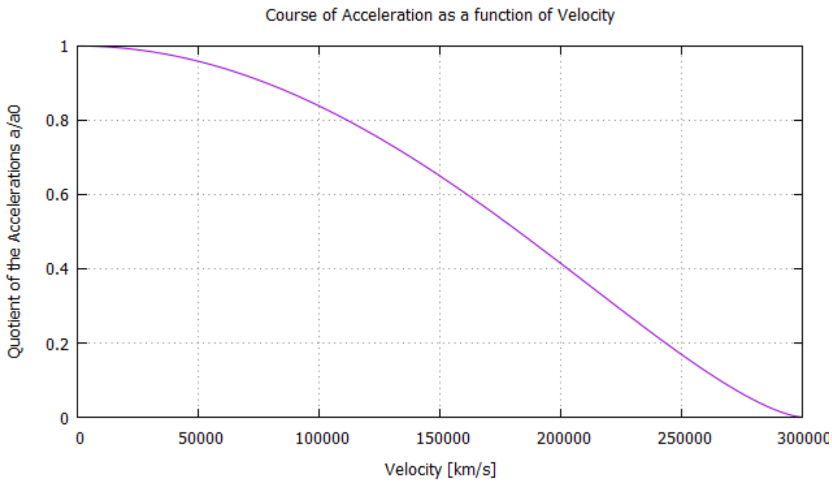

Fig. 25

Equation (16.5) shows that with constant force and increasing speed, the acceleration gradually decreases and tends towards zero near the speed of light (Fig. 25).

This result agrees with the experimental observations that can be made in the particle accelerators.

It is interesting to note that the relation (16.5) can also be derived from the differential equation (6.3) used for the calculation of kinetic energy in chapter 6:

$$Fds = \frac{m_0 v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}dv \qquad (6.3)$$

The relation (16.5) is obtained with the following steps:

(i) In (6.3) replace the infinitesimal distance ds by the product of the velocity with the time differential $\boldsymbol{vdt}$

(ii) replace the differential of the speed $\boldsymbol{dv}$ with the product of the acceleration with the time differential $\boldsymbol{adt}$,

(iii) shortening once and

(iv) finally, solve the equation for acceleration $\boldsymbol{a}$.

Case 2. Vector calculation.

If the path does not run in the same direction of the force, the relation (5.1) for the infinitesimal work must be written as follows:

$$\vec{F} \cdot d\vec{s} = \vec{F} \cdot \vec{v}dt = c^2 dm \qquad (16.6)$$

Where $\vec{F} \cdot d\vec{s}$ represents the scalar product of the force with the infinitesimal displacement vector.

From (16.6) it follows:

$$\frac{dm}{dt} = \frac{\vec{F} \cdot \vec{v}}{c^2} \qquad (16.7)$$

Relation (16.7) and the relativistic mass formula $m = m_0/\sqrt{1 - v^2/c^2}$ are now inserted in (16.1):

$$\vec{F} = \frac{\vec{F} \cdot \vec{v}}{c^2}\vec{v} + \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\frac{d\vec{v}}{dt} \qquad (16.8)$$

In the following vector calculation, we will use the parallel component (longitudinal component) and the perpendicular component (transverse component) to the velocity $\vec{v}$ for all vectors:

$$\vec{F} = \begin{pmatrix} F_L \\ F_T \end{pmatrix}; \qquad \frac{d\vec{v}}{dt} = \vec{a} = \begin{pmatrix} a_L \\ a_T \end{pmatrix}; \qquad \vec{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

Equation (16.8) leads to:

$$\begin{pmatrix} F_L \\ F_T \end{pmatrix} = \frac{1}{c^2}\left[\begin{pmatrix} F_L \\ F_T \end{pmatrix} \cdot \begin{pmatrix} v \\ 0 \end{pmatrix}\right]\begin{pmatrix} v \\ 0 \end{pmatrix} + \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\begin{pmatrix} a_L \\ a_T \end{pmatrix} \qquad \Rightarrow$$

$$\begin{pmatrix} \boldsymbol{F_L} \\ \boldsymbol{F_T} \end{pmatrix} - \frac{\boldsymbol{F_L v}}{\boldsymbol{c^2}} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{pmatrix} = \frac{\boldsymbol{m_0}}{\sqrt{\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}} \begin{pmatrix} \boldsymbol{a_L} \\ \boldsymbol{a_T} \end{pmatrix} \quad \Rightarrow$$

Since $\boldsymbol{F_L} \,||\, \boldsymbol{v}$ follows:

$$\begin{pmatrix} \boldsymbol{F_L} \\ \boldsymbol{F_T} \end{pmatrix} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}} \begin{pmatrix} \boldsymbol{F_L} \\ \boldsymbol{0} \end{pmatrix} = \frac{\boldsymbol{m_0}}{\sqrt{\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}} \begin{pmatrix} \boldsymbol{a_L} \\ \boldsymbol{a_T} \end{pmatrix} \quad \Rightarrow$$

$$\begin{pmatrix} (\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}})\boldsymbol{F_L} \\ \boldsymbol{F_T} \end{pmatrix} = \frac{\boldsymbol{m_0}}{\sqrt{\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}}} \begin{pmatrix} \boldsymbol{a_L} \\ \boldsymbol{a_T} \end{pmatrix} \quad \Rightarrow$$

From this it follows for the longitudinal and the transversal component of the acceleration:

$$\boldsymbol{a_L} = \frac{\boldsymbol{F_L}}{\boldsymbol{m_0}} \left(\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}\right)^{\frac{3}{2}} \quad ; \quad \boldsymbol{a_T} = \frac{\boldsymbol{F_T}}{\boldsymbol{m_0}} \left(\boldsymbol{1} - \frac{\boldsymbol{v^2}}{\boldsymbol{c^2}}\right)^{\frac{1}{2}} \qquad (16.9)$$

Figure 26 shows the curves of the longitudinal and the transverse vector components of the relativistic acceleration as a function of the speed. These relations also agree with those that were derived with the "mathematical formalism of the theory of relativity".

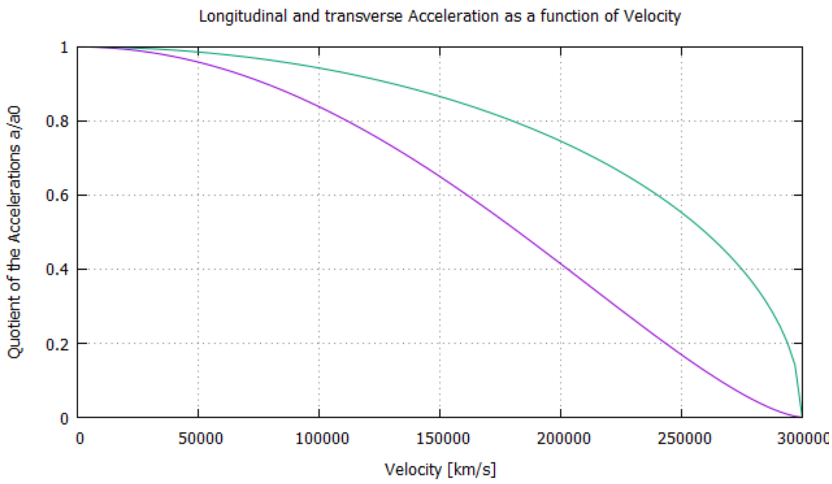

Fig. 26

The definition of the second law of dynamics in its general form makes it possible to derive the expression of acceleration as a function of speed. The curves in Figures show that for velocities close to the speed of light, the acceleration approaches zero, as confirmed by experimental observations.