Appendix

An elementary derivation of $E=mc^2$

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A I.

(Received 6 March 1989; accepted for publication 12 April 1989)

The equality $E = mc^2$ is derived in a fashion suitable for presentation in an elementary physics course for nonscience majors. It assumes only 19th-century physics and knowledge of the photon.

Einstein's original derivation of the relation between the inertia and the energy content of a body[1] assumes the knowledge of the relativistic Doppler effect. It was done after his seminal paper on special relativity, but 17 years before Compton's experiments of 1922. Had it been done before 1905 and had the particle properties of the photon already been known at that time, the following derivation could conceivably have been carried out.

One starts with the following four simple assumptions.

(1) The Newtonian formulas for the kinetic energy and the linear momentum of a free body of mass m and speed v, $mv^2/2$ and mv. Correspondingly, one assumes $(v/c)^2 \ll 1$ throughout.

(2) The laws of conservation of energy and momentum, but not the law of conservation of mass believed before 1905 to be valid.

(3) The Doppler effect, which has been known since the first half of the 19th century: Radiation (whether it be sound waves or electromagnetic waves) of speed c and frequency ν (when the observer is at rest relative to the source) will be perceived to have that frequency altered by a factor $1 + v/c$ $(1 - v/c)$ when the observer moves relative to the source with speed v and in a direction toward it (away from it).

(4) Electromagnetic radiation (in particular visible light) as produced by a source at rest consists of quanta (photons) that have particle properties: Radiation of frequency ν consists of photons of energy $h\nu$ and momentum $h\nu/c$. h and c are *constants*.

If these properties of photons had been known to a 19th-century physicist, these statements would have been the accepted truth of the day. Accept them therefore as the basis for an analysis of the following physical process that could be considered as a thought experiment, but which is not beyond realization.

A source of radiation emits two photons simultaneously while remaining at rest in some (Newtonian!) inertial reference frame R. Conservation of momentum requires these two photons to have equal and opposite momenta, and therefore equal frequencies ν. Therefore, they also have equal energies $h\nu$. Conservation of energy requires that the internal energy of the source diminishes by an amount

$$\Delta E = 2h\nu. \qquad (1)$$

Assume now that this process is viewed from a different reference frame R', which is moving uniformly relative to the rest frame of the source and in such a way that the source is seen to move with speed v in the same direction as one of the photons. Conservation of momentum then requires the momentum of the source before emission p'_i to be equal to the momentum of the source after emission p'_f together with the two momenta of the photons:

$$p'_i = p'_f + \left(\frac{h\nu}{c}\right)\left(1 + \frac{v}{c}\right) - \left(\frac{h\nu}{c}\right)\left(1 - \frac{v}{c}\right).$$

The loss in source momentum, $\Delta p'$, is therefore

$$\Delta p' = (2h\nu/c^2)v. \qquad (2)$$

But the source in reference frame R is at rest both before and after emission; in frame R' it must therefore have the same speed v both before and after emission. Now, according to assumption (1) above, the Newtonian formula for momentum is the product of mass times speed. The momentum loss of the source is thus found to *require* a change in mass Δm times v; and that change of mass is found to be

$$\Delta m = 2h\nu/c^2. \qquad (3)$$

Conservation of energy further requires that the initial energy of the source E'_i be equal to its final energy E'_f, together with the energies of the two photons:

$$E'_i = E'_f + h\nu(1 + v/c) + h\nu(1 - v/c),$$

or

$$\Delta E' = 2h\nu = \Delta E. \qquad (4)$$

Thus the energy loss ΔE of the source is the same in both reference frames, R and R'.

Inserting Eq. (4) into Eq. (3), the change of mass is found to be

$$\Delta m = \Delta E/c^2. \qquad (5)$$

One is thus forced to conclude that the emission of the two photons reduces the mass of the source, and this mass loss amounts to an energy loss of $\Delta E = \Delta mc^2$. The equivalence of inertial mass loss and energy loss has thus been derived from the above four assumptions.

One can go one step further and assume that *all* of the mass of the source is used up by emitting photons of large enough frequency; it must be so that $h\nu = m_i c^2/2$, where m_i is the initial mass of the source. That mass then disappears, and its energy is present in the two photons that have total energy $E = m_i c^2$. Therefore, the mass m_i must have been associated with that amount of energy.

This concludes the elementary derivation. One can add to it several layers of sophistication. The simplest one is to permit the source (as seen from R') to move at an arbitrary angle α relative to one of the photons. This adds a factor $\cos\alpha$ in the Doppler effect. Momentum conservation in R' then requires two equations, one for the parallel and one for the perpendicular components of the momenta. The end result is of course the same.

Another modification would be to assume the relativistic Doppler effect and the relativistic expressions for the linear momentum of the source. One can still maintain $\alpha = 0$ at first. This results in the relativistic relation between the (rest) mass and the total energy (mass energy plus kinetic energy) of the source. If a finite angle α is also assumed, one returns to the assumptions underlying Einstein's original derivation, and our assumption (4) is no longer necessary to derive Eq. (5).

This article was motivated by the criticism[2] of a faulty derivation in my recent book.[3]

A II. (Referenced in chapter 2 and 13)

The mass formula $\boldsymbol{m = m_0/\sqrt{1-\frac{v^2}{c^2}}}$ is used in the equation $\boldsymbol{F = d(mv)/dt}$ of the second law of dynamics. The differentiation leads directly to the formula of relativistic acceleration for arbitrary velocities. Thus, the assertion is refuted that the second law of dynamics can only be used with constant mass.

$$\boldsymbol{F = m_0\frac{d}{dt}\,v\left(1-\frac{v^2}{c^2}\right)^{-0.5}}$$

$$\boldsymbol{F = m_0\frac{dv}{dt}\left(1-\frac{v^2}{c^2}\right)^{-0.5} - 0.5m_0v\left(1-\frac{v^2}{c^2}\right)^{-1.5}(-2\frac{v}{c^2})\frac{dv}{dt}}$$

$$\boldsymbol{F = m_0\left(1-\frac{v^2}{c^2}\right)^{-0.5}\frac{dv}{dt} + m_0\left(1-\frac{v^2}{c^2}\right)^{-0.5}\left(1-\frac{v^2}{c^2}\right)^{-1.0}\frac{v^2}{c^2}\frac{dv}{dt}}$$

$$\boldsymbol{F = m_0\left(1-\frac{v^2}{c^2}\right)^{-0.5}\left(1+\left(1-\frac{v^2}{c^2}\right)^{-1.0}\frac{v^2}{c^2}\right)\frac{dv}{dt}}$$

$$\boldsymbol{F = m_0\left(1-\frac{v^2}{c^2}\right)^{-0.5}\left(1+\frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)\frac{dv}{dt}}$$

$$\boldsymbol{F = m_0\left(1-\frac{v^2}{c^2}\right)^{-1.5}\frac{dv}{dt}}$$

$$\boldsymbol{a = \frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

A III. (Referenced in chapter 8)

For the thought experiment described in Figure 9, a further derivation is performed here which uses the law of conservation of momentum.

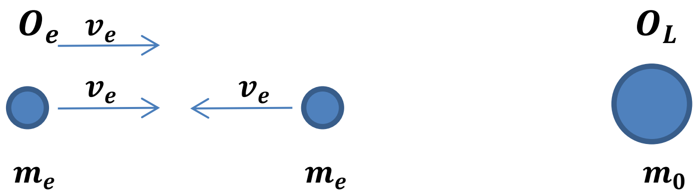


Fig. 9

$$\frac{m_{0e}v_{ee}}{\sqrt{1-\frac{v_{ee}^2}{c^2}}} = \frac{2m_{0e}v_e}{1-\frac{v_e^2}{c^2}} \qquad \Rightarrow$$

$$\frac{v_{ee}^2}{1-\frac{v_{ee}^2}{c^2}} = 4\frac{v_e^2}{1-2\frac{v_e^2}{c^2}+\frac{v_e^4}{c^4}} \qquad \Rightarrow$$

$$v_{ee}^2 - 2\frac{v_{ee}^2v_e^2}{c^2} + \frac{v_{ee}^2v_e^4}{c^4} = 4v_e^2 - 4\frac{v_{ee}^2v_e^2}{c^2} \qquad \Rightarrow$$

$$v_{ee}^2 + 2\frac{v_{ee}^2v_e^2}{c^2} + \frac{v_{ee}^2v_e^4}{c^4} = 4v_e^2 \qquad \Rightarrow$$

$$v_{ee} + \frac{v_{ee}v_e^2}{c^2} = 2v_e \qquad \Rightarrow$$

$$v_{ee} = \frac{2v_e}{1+\frac{v_e^2}{c^2}} \qquad (8.6)$$

A IV. (Referenced in chapter 10)

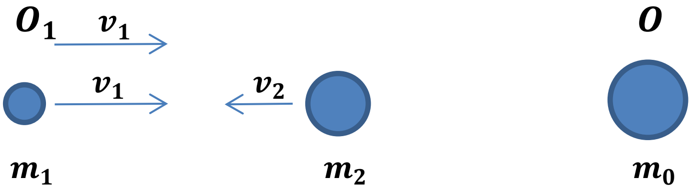


Fig. 14

For the thought experiment described in Figure 14, a further derivation is performed here which uses the law of conservation of energy. From the point of view of observer $\boldsymbol{O_1}$ applies**:**

- $\boldsymbol{m_1}$ is in rest,
- $\boldsymbol{m_2}$ moves with relative velocity $\boldsymbol{v_{12}}$ which is derived here for arbitrary velocities (for low velocities as already known $v_{12} = v_1 + v_2$ is valid),
- $\boldsymbol{m_0}$ moves at the speed $\boldsymbol{v_1}$.

That's why we can start with the following equation:

$$m_{01}c^2 + \frac{m_{02}c^2}{\sqrt{1-\beta_{12}^2}} = \frac{m_0c^2}{\sqrt{1-\beta_1^2}} \qquad \text{(AIV.1)}$$

The terms for $\boldsymbol{m_{01}}$ from equation (10.3)

$$\frac{m_{01}}{\sqrt{1-\beta_1^2}} = \frac{m_{02}}{\sqrt{1-\beta_2^2}}\frac{\beta_2}{\beta_1} \qquad (10.3)$$

and for $\boldsymbol{m_0}$ from equation (10.4)

$$m_0 = \frac{m_{02}}{\sqrt{1-\beta_2^2}}\left(1+\frac{\beta_2}{\beta_1}\right) \qquad (10.4)$$

are inserted into equation (AIV.1):

$$\frac{\cancel{m_{02}}}{\sqrt{1-\beta_2^2}}\frac{\beta_2}{\beta_1}\sqrt{1-\beta_1^2} + \frac{\cancel{m_{02}}}{\sqrt{1-\beta_{12}^2}} = \frac{\cancel{m_{02}}}{\sqrt{1-\beta_2^2}\sqrt{1-\beta_1^2}}\left(1+\frac{\beta_2}{\beta_1}\right)$$

$$\frac{\frac{\beta_2}{\beta_1}\sqrt{1-\beta_1^2}\sqrt{1-\beta_{12}^2} + \sqrt{1-\beta_2^2}}{\cancel{\sqrt{1-\beta_2^2}}\sqrt{1-\beta_{12}^2}} = \frac{\left(1+\frac{\beta_2}{\beta_1}\right)}{\cancel{\sqrt{1-\beta_2^2}}\sqrt{1-\beta_1^2}}$$

$$\frac{\beta_2}{\beta_1}\left(1-\beta_1^2\right)\sqrt{1-\beta_{12}^2} + \sqrt{1-\beta_2^2}\sqrt{1-\beta_1^2} = \sqrt{1-\beta_{12}^2} + \frac{\beta_2}{\beta_1}\sqrt{1-\beta_{12}^2}$$

$$\cancel{\frac{\beta_2}{\beta_1}\sqrt{1-\beta_{12}^2}} - \beta_1\beta_2\sqrt{1-\beta_{12}^2} + \sqrt{1-\beta_2^2}\sqrt{1-\beta_1^2} = \sqrt{1-\beta_{12}^2} + \cancel{\frac{\beta_2}{\beta_1}\sqrt{1-\beta_{12}^2}}$$

$$\sqrt{1-\beta_2^2}\sqrt{1-\beta_1^2} = (1+\beta_1\beta_2)\sqrt{1-\beta_{12}^2}$$

$$(1-\beta_2^2)(1-\beta_1^2) = (1+\beta_1\beta_2)^2(1-\beta_{12}^2)$$

$$\cancel{1} - \beta_1^2 - \beta_2^2 + \cancel{\beta_1^2\beta_2^2} = \cancel{1} + 2\beta_1\beta_2 + \cancel{\beta_1^2\beta_2^2} - (1+\beta_1\beta_2)^2\beta_{12}^2$$

$$-\beta_1^2 - \beta_2^2 = 2\beta_1\beta_2 - (1+\beta_1\beta_2)^2\beta_{12}^2$$

$$(1+\beta_1\beta_2)^2\beta_{12}^2 = (\beta_1+\beta_2)^2$$

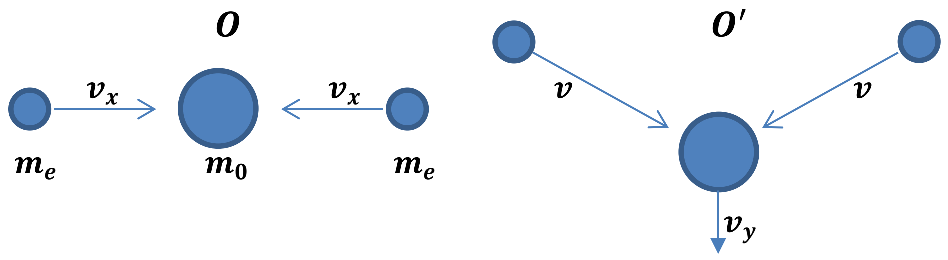
$$\beta_{12} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

$$v_{12} = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}} \tag{10.6}$$

Q.E.D.

A V. (Referenced in chapter 11)

Speed from the perspective of an orthogonal moving observer.



We imagine the central collision between two electrons which move towards each other with equal velocities v_x, as illustrated on the left in Figure.

Suppose that as a result of the collision, a new particle of mass m_0 is formed. This is from the perspective of an observer O in the origin of a coordinate system resting on him.

Let us now consider the same thought experiment from the point of view of a second observer O' moving upwards at the velocity v_y orthogonal to the direction of motion of the colliding electrons (see Figure, right).

From O point of view, the following applies because of the law of conservation of energy (see Figure, left):

$$m_0c^2 = \frac{2m_{0e}c^2}{\sqrt{1-\frac{v_x^2}{c^2}}} \qquad (\mathrm{V.1})$$

From observer O' point of view, the formed particle moves further down along the Y-axis after the collision with velocity v_y (see right in Figure) and for him the result is:

$$\frac{m_0c^2}{\sqrt{1-\frac{v_y^2}{c^2}}} = \frac{2m_{0e}c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (\mathrm{V.2})$$

By using (V.1) in (V.2) we get:

$$\frac{2m_{0e}c^2}{\sqrt{1-\frac{v_x^2}{c^2}}\sqrt{1-\frac{v_y^2}{c^2}}} = \frac{2m_{0e}c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad ==>$$

$$\left(1-\frac{v_x^2}{c^2}\right)\left(1-\frac{v_y^2}{c^2}\right) = 1-\frac{v^2}{c^2} \qquad ==>$$

$$1-\frac{v_x^2}{c^2}-\frac{v_y^2}{c^2}+\frac{v_x^2v_y^2}{c^4} = 1-\frac{v^2}{c^2}$$

This leads to:

$$\boldsymbol{v^2 = v_x^2 + v_y^2 - \frac{v_x^2 v_y^2}{c^2}} \qquad (\mathrm{V.3})$$

It is easy to see that the relation (V.3) for $\boldsymbol{v_x} \ll \boldsymbol{c}$ and $\boldsymbol{v_y} \ll \boldsymbol{c}$ is reduced to $\boldsymbol{v^2 = v_x^2 + v_y^2}$, as is known in the context of the addition of orthogonal vectors.

If one or both components are equal to $\boldsymbol{c}$, then it results: $\boldsymbol{v = c}$.

For other arbitrary values of v_x and v_y, v never exceeds the speed of light according to the relation (V.3), as it is to be expected within the framework of the theory of relativity